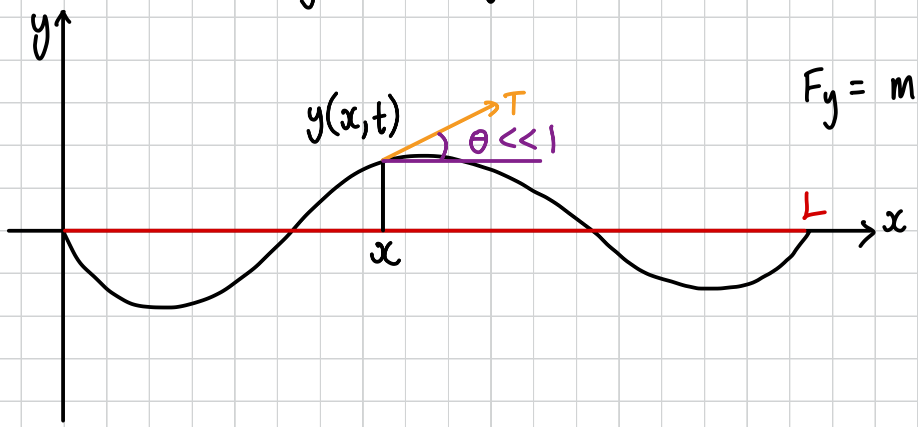
# Wave Mechanics

## 1. Waves on a String

### Deriving Wave Equation in 1D

Suppose we have a string with length $L$, uniform linear density $\rho$, tension $T$



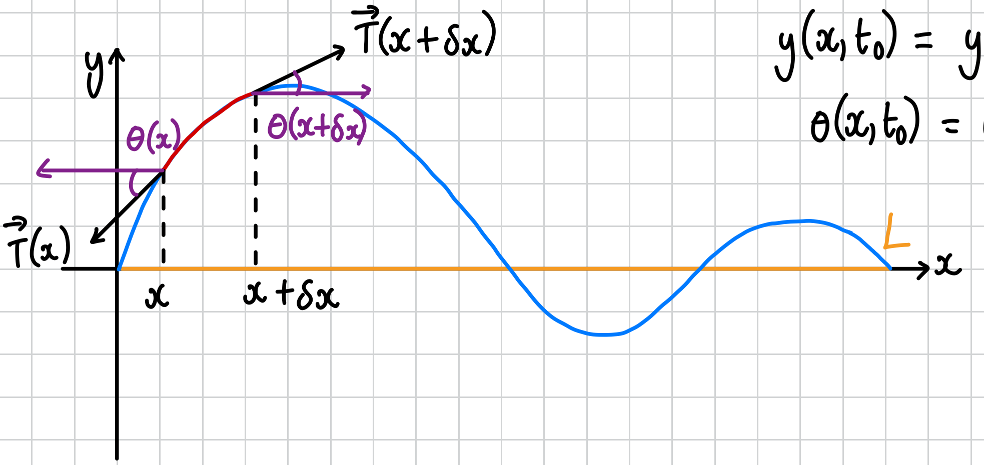

$$F_y = ma = m\frac{\partial^2 y}{\partial t^2}(x,t)$$

Assumptions:

1) No gravity, no drag due to air friction
2) Only one force acting on system; Tension $\Longrightarrow$ equilibrium position is straight
3) Continuum approximation $\Longrightarrow$ string is continuous
4) Amplitude not excessively large $\Longrightarrow$ $\theta \ll 1$ (small)
5) Can only vibrate in vertical, no horizontal motion

Tension Forces

Consider the following diagram at specific time $t = t_0$

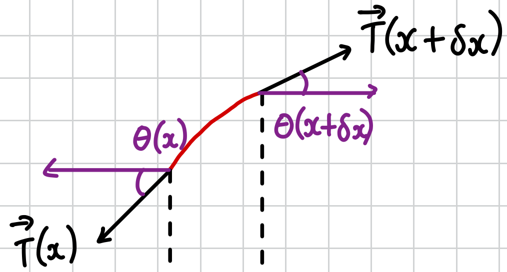



$$y(x, t_0) = y(x)$$

$$\theta(x, t_0) = \theta(x)$$

Isolate a small section where

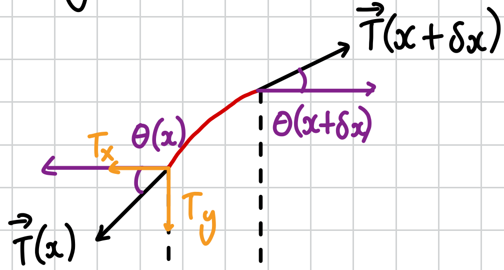
- $0 < x < L$,
- $\delta x \ll 1$ (small)




- Magnitude of Tension Force

$$|\vec{T}(x)| = x$$

**Projections:** Decomposing the tension force into horizontal and vertical components,




From the diagram:

$$\vec{T}(x) = |\vec{T}(x)| \begin{pmatrix} -\cos\theta(x) \\ -\sin\theta(x) \end{pmatrix} \Rightarrow \boxed{\vec{T} = T\begin{pmatrix} -\cos\theta(x) \\ -\sin\theta(x) \end{pmatrix}}$$

Similarly $\boxed{\vec{T}(x+\delta x) = T\begin{pmatrix} \cos\theta(x+\delta x) \\ \sin\theta(x+\delta x) \end{pmatrix}}$

The **total force** is

$$\vec{F} = \vec{T}(x) + \vec{T}(x+\delta x)$$

$$\Rightarrow \boxed{\vec{F} = \begin{pmatrix} \cos\theta(x+\delta x) - \cos\theta(x) \\ \sin\theta(x+\delta x) - \sin\theta(x) \end{pmatrix}}$$

Assuming $\delta x << 1$ (small) by **Taylor's thm**

$$\theta(x+\delta x) \sim \theta(x) + \delta x \frac{\partial\theta(x)}{\partial x} + O(\delta x^2)$$

and therefore

$$\vec{F} = T\begin{pmatrix} \cos\left(\theta(x) + \delta x \frac{\partial\theta(x)}{\partial x} + O(\delta x^2)\right) - \cos\theta(x) \\ \sin\left(\theta(x) + \delta x \frac{\partial\theta(x)}{\partial x} + O(\delta x^2)\right) - \sin\theta(x) \end{pmatrix}$$

Again, applying Taylor's thm;

- $$\cos\left(\theta(x) + \delta x \frac{\partial \theta}{\partial x} + O(\delta x^2)\right) = \cos\theta(x) + \delta x \frac{\partial \theta}{\partial x} \frac{d}{d\theta(x)} \cos(\theta(x)) + O(\delta x^2)$$

$$= \cos\theta(x) - \delta x \frac{\partial \theta}{\partial x} \sin\theta(x) + O(\delta x^2)$$

- $$\sin\left(\theta(x) + \delta x \frac{\partial \theta}{\partial x} + O(\delta x^2)\right) = \sin\theta(x) + \delta x \frac{\partial \theta}{\partial x} \frac{d}{d\theta(x)} \sin(\theta(x)) + O(\delta x^2)$$

$$= \sin\theta(x) + \delta x \frac{\partial \theta}{\partial x} \cos\theta(x) + O(\delta x^2)$$

Substituting gives

$$\boxed{\vec{F} = T\delta x \frac{\partial \theta}{\partial x} \begin{pmatrix} -\sin\theta(x) \\ \cos\theta(x) \end{pmatrix} + O(\delta x^2)}$$

Since oscillations are small,

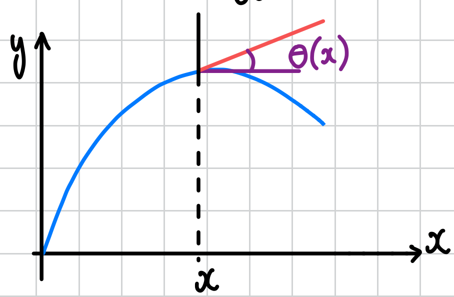
$\theta(x)$ is small $\Rightarrow \theta \ll 1$

$$\Rightarrow \sin\theta(x) \sim \theta(x) + O(\theta(x)^2) = O(\theta)$$

$$\cos\theta(x) \sim 1 + O(\theta(x)^2)$$

$$\Rightarrow \boxed{\vec{F} = T\delta x \frac{\partial \theta}{\partial x} \begin{pmatrix} 0 \\ 1 \end{pmatrix} + O(\delta x^2) + O(\theta)}$$

Also need to relate $\frac{\partial \theta(x)}{\partial x}$ to shape of $y(x)$




$$\frac{\partial y}{\partial x} = \tan\theta(x)$$

$$\frac{\partial y}{\partial x} = \tan\theta(x) \Rightarrow \frac{\partial^2 y}{\partial x^2} = \frac{1}{\cos^2\theta} \frac{\partial}{\partial x} \theta(x) \quad \text{Chain rule}$$

$$\Rightarrow \frac{\partial^2 y}{\partial x^2} \sim \frac{\partial}{\partial x} \theta(x) + O(\theta^2) \quad \theta \text{ is small}$$

Putting it all together we get

$$\vec{F} = \begin{pmatrix} F_x \\ F_y \end{pmatrix} = \left[ T \frac{\partial^2}{\partial x^2} y(x) \begin{pmatrix} 0 \\ 1 \end{pmatrix} + \mathcal{O}(\theta(x)) \right] \delta x + \mathcal{O}(\delta x^2)$$

$$\left.\begin{aligned} \Longrightarrow \quad & F_x = 0 + \mathcal{O}(\theta(x)) + \mathcal{O}(\delta x^2) \\ & F_y = \left[ T \frac{\partial^2}{\partial x^2} y(x) + \mathcal{O}(\theta(x)) \right] \delta x + \mathcal{O}(\delta x^2) \end{aligned}\right\} \quad (*)$$

Note $\rho = \frac{m}{L} \Longrightarrow m = \delta x \rho$ assume $\rho$ constant

## Applying Newton's law

Newton's second law

$$m\vec{a}(x,t) = \vec{F}(x,t)$$

Assuming only vertical vibrations $\Longrightarrow$ Horizontal forces 0

$$\Longrightarrow F_x = 0$$

By Newton's second law,

$$F_y = m a_y(x,t) = m \frac{\partial^2}{\partial t^2} y(x,t)$$

Equating to $(*)$

$$\rho \cancel{\delta x} \frac{\partial^2}{\partial t^2} y(x,t) = F_y = \left[ T \frac{\partial^2}{\partial x^2} y(x) + \mathcal{O}(\theta(x)) \right] \cancel{\delta x} + \mathcal{O}(\delta x^2)$$

$$\Longrightarrow \quad \rho \frac{\partial^2}{\partial t^2} y(x,t) = T \frac{\partial^2}{\partial x^2} y(x) + \mathcal{O}(\theta(x)) + \mathcal{O}(\delta x^2)$$

$$\Longrightarrow \quad \rho \frac{\partial^2}{\partial t^2} y(x,t) = T \frac{\partial^2}{\partial x^2} y(x) + \mathcal{O}(\delta x^2)$$ Dropping $\mathcal{O}(\theta(x))$ terms

$$\Big\downarrow \ \delta x \to 0$$

**1D Wave equation**

$$\boxed{\frac{\partial^2}{\partial t^2} y(x,t) = c^2 \frac{\partial^2 y}{\partial x^2} y(x,t)}$$

Also written as

$$\boxed{\partial_t^2 y(x,t) = c^2 \partial_x^2 y(x,t)}$$ 1D WAVE EQUATION

where

$$\boxed{c^2 = \frac{T}{\rho}}$$ Wave Velocity

## Dimensional Analysis

$$\left[\frac{\partial^2 y(x,t)}{\partial t^2}\right] = [c^2]\left[\frac{\partial^2 y(x,t)}{\partial x^2}\right] \implies \frac{L}{T^2} = [c]^2 \frac{1}{L}$$

$$\implies \boxed{[c] = \frac{L}{T}}$$

Checking that this agrees with dimensions of wave velocity defn;

$$[\rho] = \frac{M}{L} \implies [c] = \sqrt{\frac{[T]}{[\rho]}} = \sqrt{\frac{ML/T^2}{M/L}} = \sqrt{\frac{L^2}{T^2}} = \frac{L}{T} \checkmark$$

# The solution of d'Alembert

We will first solve the 1D-Wave Equation ignoring boundary conditions

Consider the 1D-Wave Eqn:

$$\boxed{\partial_t^2 y(x,t) = c^2 \partial_x^2 y(x,t)}$$ 1D WAVE EQUATION

## Change of Co-ordinates

Using appropriate co-ordinate transformation;

$$(x,t) \longrightarrow (\xi(x,t), \eta(x,t))$$

our function $y$ becomes

$$y(x,t) \equiv \tilde{y}(\xi(x,t), \eta(x,t)) \qquad \forall x,t$$

We want to transform wave eqn

$$\partial_t^2 y(x,t) = c^2 \partial_x^2 y(x,t) \longrightarrow \boxed{\partial_\xi \partial_\eta \tilde{y}(\xi,\eta) = \ell(\tilde{y})}$$ canonical form

Use change of co-ordinates

$$\begin{cases} \xi(x,t) = x + ct \\ \eta(x,t) = x - ct \end{cases}$$

Finding derivatives using chain rule

i) $$\partial_t \tilde{y}(\xi(x,t),\eta(x,t)) = \frac{\partial \tilde{y}}{\partial \xi}\frac{\partial \xi}{\partial t} + \frac{\partial \tilde{y}}{\partial \eta}\frac{\partial \eta}{\partial t}$$

$$= \partial_\xi \tilde{y}\, c - \partial_\eta \tilde{y}\, c$$

$$\Rightarrow \boxed{\partial_t \tilde{y}(\xi(x,t),\eta(x,t)) = \partial_\xi \tilde{y}\, c - \partial_\eta \tilde{y}\, c}$$

ii) $$\partial_x \tilde{y}(\xi(x,t),\eta(x,t)) = \frac{\partial \tilde{y}}{\partial \xi}\frac{\partial \xi}{\partial x} + \frac{\partial \tilde{y}}{\partial \eta}\frac{\partial \eta}{\partial x}$$

$$= \partial_\xi \tilde{y} + \partial_\eta \tilde{y}$$

$$\Rightarrow \boxed{\partial_x \tilde{y}(\xi(x,t),\eta(x,t)) = \partial_\xi \tilde{y} + \partial_\eta \tilde{y}}$$

Finding second derivative,

i) $$\partial_t^2 \tilde{y}(\xi(x,t),\eta(x,t)) = \partial_t(\partial_\xi \tilde{y}\, c - \partial_\eta \tilde{y}\, c)$$

$$= \partial_t \partial_\xi \tilde{y}\, c - \partial_t \partial_\eta \tilde{y}\, c$$

$$= \partial_\xi \partial_t \tilde{y}\, c - \partial_\eta \partial_t \tilde{y}\, c \qquad \text{Clairaut's Thm}$$

$$= [\partial_\xi(\partial_\xi \tilde{y} - \partial_\eta \tilde{y}) - \partial_\eta(\partial_\xi \tilde{y} - \partial_\eta \tilde{y})]c^2$$

$$= [\partial_\xi^2 \tilde{y} + \partial_\eta^2 \tilde{y} - 2\partial_\xi \partial_\eta \tilde{y}]c^2$$

$$\Rightarrow \boxed{\partial_t^2 \tilde{y} = [\partial_\xi^2 \tilde{y} + \partial_\eta^2 \tilde{y} - 2\partial_\xi \partial_\eta \tilde{y}]c^2}$$

ii) $$\partial_x^2 \tilde{y}(\xi(x,t),\eta(x,t)) = \partial_x(\partial_\xi \tilde{y} + \partial_\eta \tilde{y})$$

$$= \partial_x \partial_\xi \tilde{y} + \partial_x \partial_\eta \tilde{y}$$

$$= \partial_\xi \partial_x \tilde{y} + \partial_\eta \partial_x \tilde{y} \qquad \text{Clairaut's Thm}$$

$$= \partial_\xi(\partial_\xi \tilde{y} + \partial_\eta \tilde{y}) + \partial_\eta(\partial_\xi \tilde{y} + \partial_\eta \tilde{y})$$

$$\Rightarrow \boxed{\partial_x^2 \tilde{y} = \partial_\xi^2 \tilde{y} + \partial_\eta^2 \tilde{y} + 2\partial_\xi \partial_\eta \tilde{y}}$$

Substituting this into the 1D WAVE EQUATION gives

$$\left[\cancel{\partial_\xi^2 \tilde{y}} + \cancel{\partial_\eta^2 \tilde{y}} - 2\partial_\xi \partial_\eta \tilde{y}\right]\cancel{c^2} = \cancel{c^2}\left[\cancel{\partial_\xi^2 \tilde{y}} + \cancel{\partial_\eta^2 \tilde{y}} + 2\partial_\xi \partial_\eta \tilde{y}\right]$$

$$\Longrightarrow \boxed{\partial_\xi \partial_\eta \tilde{y}(\xi,\eta) = 0}$$ CANONICAL FORM

$$\boxed{\begin{aligned} \xi &= x + ct \\ \eta &= x - ct \end{aligned}}$$

## General Solution of wave equation

Define

$$\partial_\eta \tilde{y}(\xi,\eta) = f(\xi,\eta), \qquad f \text{ is an arbitrary function}$$

Since by canonical form,

$$\partial_\xi \partial_\eta \tilde{y}(\xi,\eta) = 0 \Rightarrow \partial_\eta \tilde{y}(\xi,\eta) = f(\xi,n) = f(\eta)$$

$$\Rightarrow \partial_\eta \tilde{y}(\xi,\eta) = f(\eta)$$

Since $f(\eta)$ is arbitrary, represent using its primitives

$$f(\eta) = \partial_\eta F(\eta)$$

Therefore we get

$$\partial_\eta \tilde{y}(\xi,\eta) = f(\eta) \Longrightarrow \partial_\eta \tilde{y}(\xi,\eta) = \partial_\eta F(\eta)$$

$$\Longrightarrow \partial_\eta\left[\tilde{y}(\xi,\eta) - F(\eta)\right] = 0$$

$$\Longrightarrow \boxed{y(\xi,\eta) = F(\eta) + C(\xi)}$$ constant of integration (1)

But we can make a similar argument for other variable

$$\partial_\xi \tilde{y}(\xi,\eta) = g(\xi,\eta) = g(\xi) = \partial_\xi G(\xi) \Rightarrow \partial_\xi \tilde{y}(\xi,\eta) = \partial_\xi G(\xi)$$

$$\Rightarrow \partial_\xi\left[\tilde{y}(\xi,\eta) - \partial_\xi G(\xi)\right]$$

$$\Rightarrow \boxed{\tilde{y}(\xi,\eta) = G(\xi) + C'(\eta)}$$ (2)

From (1) and (2), the general solution is

$$\boxed{y(\xi,\eta) = F(\eta) + G(\xi)}$$ GENERAL EQUATION OF WAVES

Therefore

$$\boxed{y(x,t) = F(x-ct) + G(x+ct)}$$ GENERAL EQUATION OF WAVES

## Travelling Waves

When $G(\xi) = 0$, the solution becomes

$$y(x,t) = F(x-ct)$$

This solution evolves by rigidly moving to the right, shape unchanged

**Right-moving wave**

$$y(x,t) = F(x-ct)$$

When $F(\eta) = 0$, the solution becomes

$$y(x,t) = G(x+ct)$$

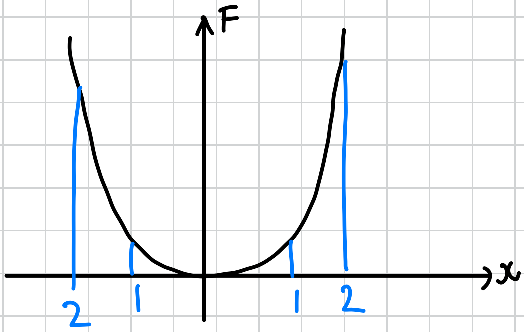
This solution evolves by rigidly moving to the left, shape unchanged

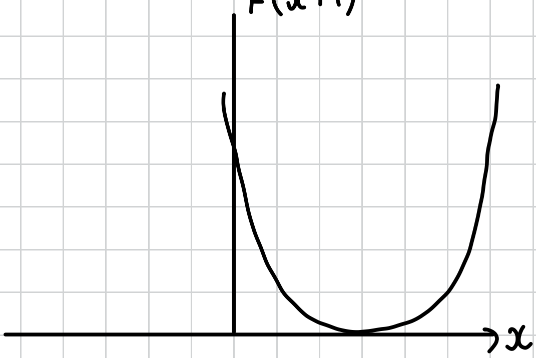
**Right-moving left**

$$y(x,t) = G(x+ct)$$

Examples:

$F(x-ct)$ , $F(x) = x^2$




A shift of one to the right

# Initial Value Problems

Finding particular solution to

$$\partial_t^2 y(x,t) = c^2 \partial_x^2 y(x,t)$$

subject to initial conditions

$$\boxed{y(x,t=0) = y_0 \qquad \partial_t y(x,t)\big|_{t=0} = v_0}$$

Substituting boundary conditions into general wave equation

$$F(x) + G(x) = y_0(x) \qquad (1)$$

$$cF'(x) - cG'(x) = -v_0(x) \qquad (2)$$

Solving (2)

$$cF'(x) - cG'(x) = c\frac{d}{dx}(F(x) - G(x)) = -v_0 \implies \frac{d}{dx}(F(x) - G(x)) = -\frac{v_0}{c}$$

$$\implies \int_0^x dF(x) - dG(x) = -\frac{1}{c}\int_0^x v_0(s)\, ds$$

$$\implies F(x) - G(x) + C = -\frac{1}{c}\int_0^x v_0(s)\, ds$$

Introduce a primitive

$$V_0(x) = \int_0^x v_0(s)\, ds - C$$

we get

$$\boxed{F(x) - G(x) = -\frac{1}{c} V_0(x)} \qquad (*)$$

Now using (*) in (1) we get

$$\boxed{\begin{aligned} F(x) &= \frac{1}{2} y_0(x) - \frac{1}{2c} V_0(x) \\ G(x) &= \frac{1}{2} y_0(x) + \frac{1}{2c} V_0(x) \end{aligned}}$$

Plugging into $y(x,t)$,

$$y(x,t) = \frac{y_0(x+ct) + y_0(x-ct)}{2} + \frac{1}{2c}\left[\int_0^{x+ct} V_0(s)\,ds - \int_0^{x-ct} V_0(s)\,ds\right]$$

C cancels

$$\Longrightarrow \quad y(x,t) = \frac{y_0(x+ct) + y_0(x-ct)}{2} + \frac{1}{2c}\left[\int_0^{x+ct} V_0(s)\,ds + \int_{x-ct}^{0} V_0(s)\,ds\right]$$

$$\Longrightarrow \quad \boxed{y(x,t) = \frac{y_0(x+ct) + y_0(x-ct)}{2} + \frac{1}{2c}\left[\int_{x-ct}^{x+ct} V_0(s)\,ds\right]}$$

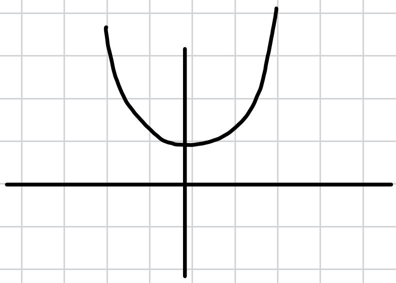
## Example

$$\begin{cases} y_0(x) = x^2 \\ V_0(x) = 0 \end{cases}$$

$$y(x,t) = \frac{(x+ct)^2 + (x-ct)^2}{2} = \frac{x^2 + c^2t^2}{2}$$

Plotting for $t = \frac{2}{c}$

$$y(x, 2/c) = \frac{x^2+4}{2}$$



# Boundaries and Interfaces

Up until now, we have considered our string to be effictively infinite. Now we add end points.

Intuitively, we know waves carry energy. They move with wave velocity $c$ therefore possess kinetic energy

In absence of dissipation energy, energy is conserved, waves cannot disappear at end of string. It must be transmitted/reflected.

**Reflection at fixed end:** Dirichlet Boundary Condition

Choose a right moving string arriving at right end of string


$y(0,t) = 0 \quad \forall t$

Not interested in places far to the left.

Mathematically considering wave on $-\infty < x < 0$

When the right end is fixed, we have the following boundary condition

$$\boxed{\begin{array}{l}\text{Dirichlet boundary condition} \\ \qquad y(0,t) = 0 \quad \forall t \in \mathbb{R}\end{array}}$$

From the general solution of a wave

$$y(x,t) = f(x-ct) + g(x+ct)$$

plug in boundary condition to get

$$0 = f(-ct) + g(ct) \implies g(ct) = -f(-ct)$$

$$\implies g(s) = -f(-s) \qquad \forall s \in \mathbb{R}$$

Thus our solution is

$$\boxed{y(x,t) = f(x-ct) - f(-x-ct) \qquad \underline{\forall x \leq 0} \quad \forall t \in \mathbb{R}}$$

solution does not exist on right plane.

This solution consists of two parts:

1) right moving part: $f(x-ct)$
2) left moving part: $-f(-x-ct)$

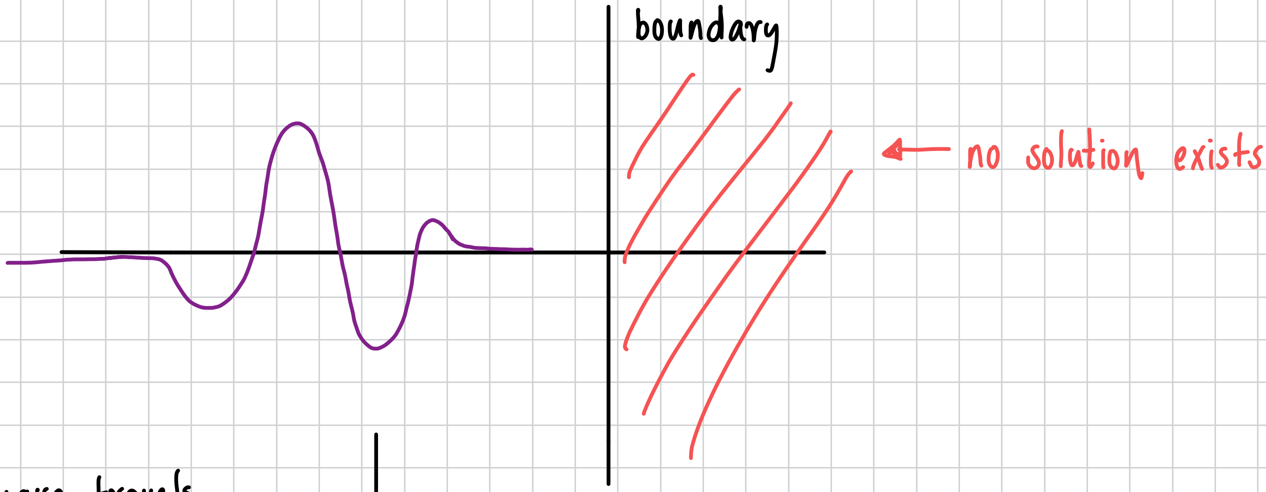
reflection x axis    reflection y axis

To understand what is happening, consider $f(x)$ to be localised around $x = -|x_0| < 0$ and typical width $\delta$

This means $f(x) \to 0$ rapidly for $x > x_0 + \delta$ and $x < x_0 - \delta$ $\left( f(x) \to 0 \text{ outside interval } (x_0 - \delta, x_0 + \delta) \right)$

Suppose at $t=0$, wave packet does not hit boundary

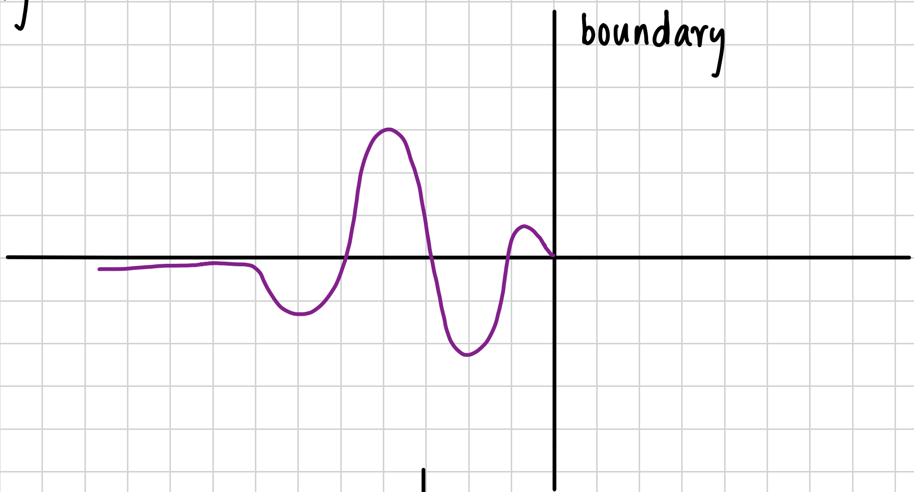
: at $t=0$

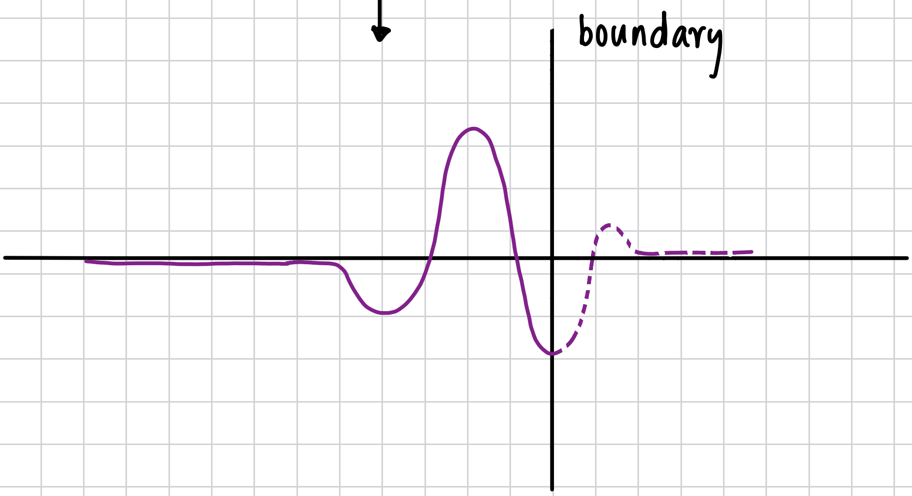


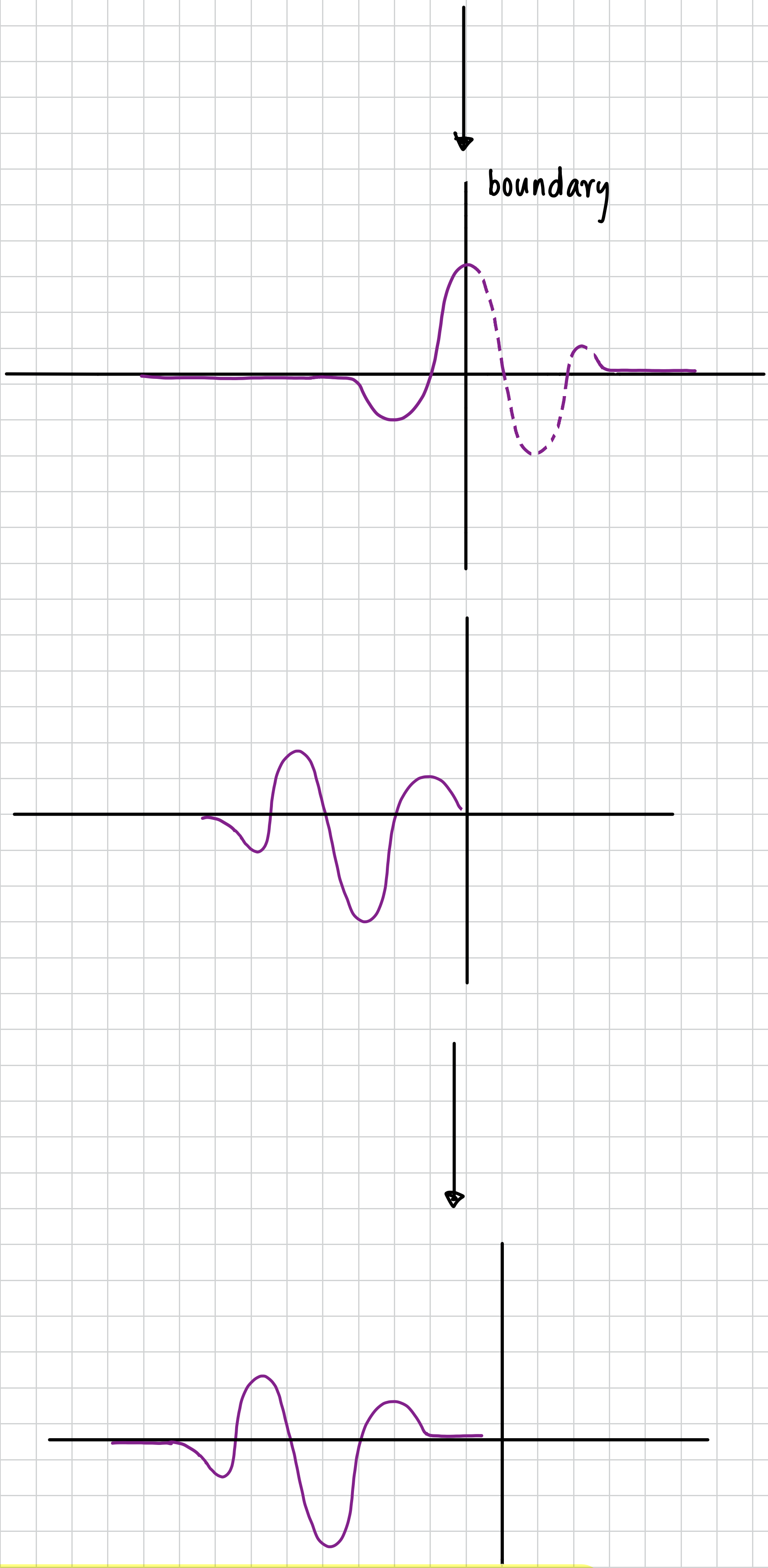

Now as the wave travels

(time evolves)

Wave hits boundary

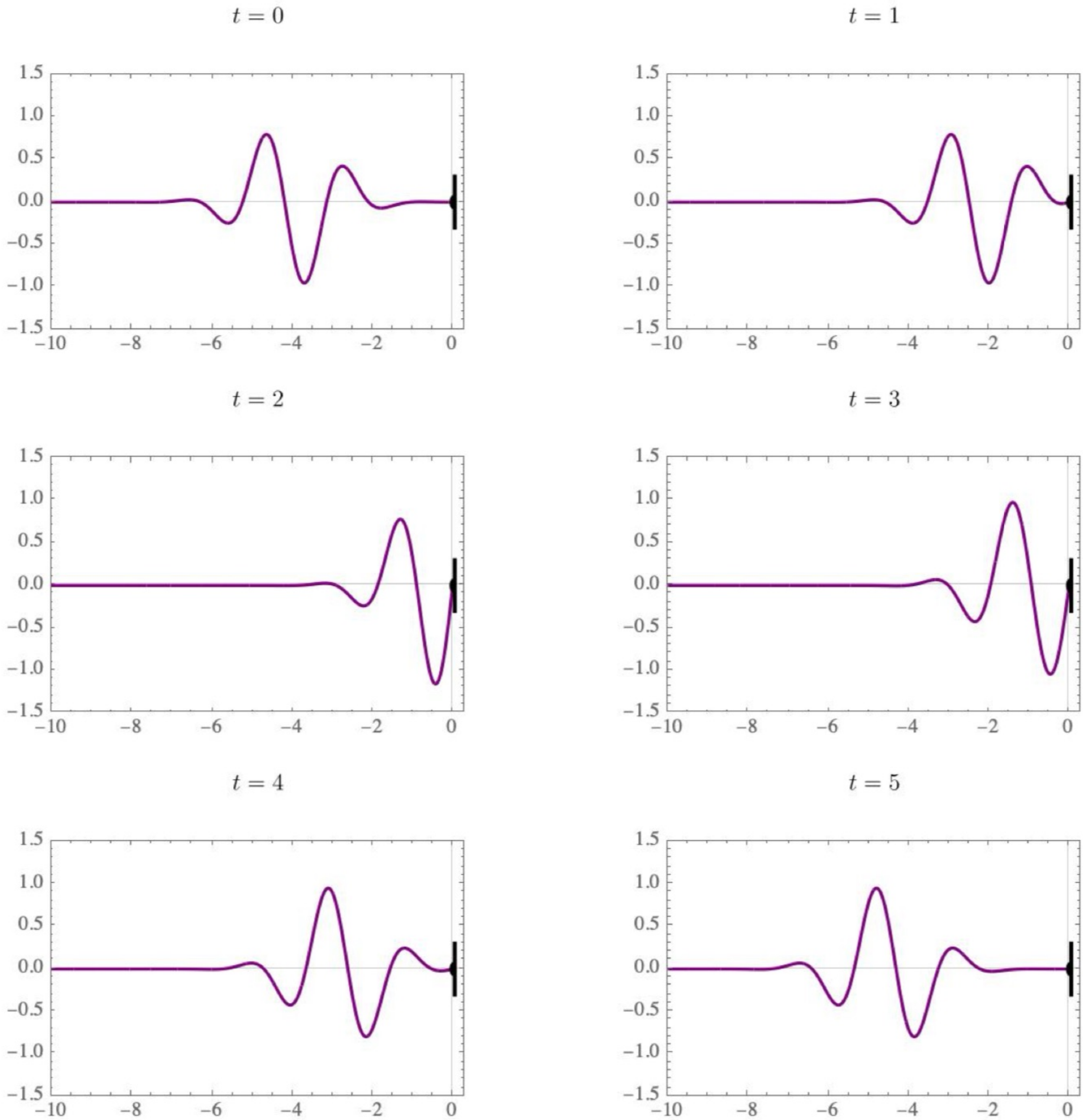





The wave is completely reflected and goes towards $-\infty$.

Below is a more accurate plot




Time snapshots of the solution to the wave equation with a Dirichlet boundary condition at $x = 0$ and $f(x) = -e^{-(x+4)^2/2}\sin(3x)$. You can see the wave packet is localised on the negative real line for $t = 0$. As time passes, it moves to the right, eventually interacting with the boundary at $x = 0$. After enough time has passed, the wave is completely reflected and travels undisturbed leftwards to $-\infty$.

## Reflections at Free End: Neumann boundary condition

Here end point is free to move.

End of string is attached to massless contraption that is free to move vertically along a rod with no friction

$x=0$

$$\partial_x y(x,t)\big|_{x=0} = 0$$

Therefore vertical component of force is 0

> **Neumann boundary condition**
>
> $$\partial_x y(x,t)\big|_{x=0} = 0 \qquad \forall t \in \mathbb{R}$$

Differentiating general wave equation $y(x,t) = f(x-ct) + g(x+ct)$

$$y'(x,t) = f'(x-ct) + g'(x+ct)$$

and plugging in boundary condition

$$f'(-ct) + g'(ct) = 0 \implies f'(s) + g'(s) = 0$$

$$\implies g'(s) = -f'(-s)$$

integrating $\implies g(s) = f(-s) + C$

Setting $C$ to 0, we get

$$\boxed{y(x,t) = f(x-ct) + f(-x-ct), \quad \forall x \leq 0, \quad \forall t \in \mathbb{R}}$$

solution does not exist on right plane

Again, we have 2 solutions:

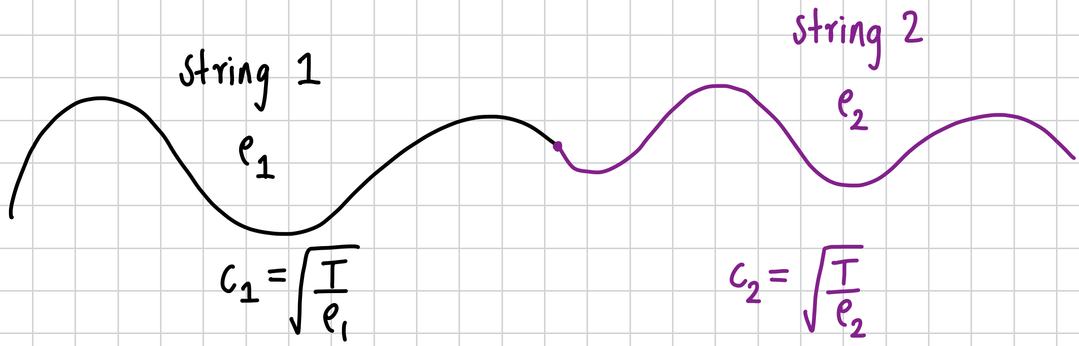
- the incoming part: $f(x-ct)$
- the reflected part: $f(-x-ct)$ $\rightarrow$ reflected front to back (vertical reflection, $y$ axis)

  $\rightarrow$ NO up-to-down reflection (no horizontal reflection, $x$ axis)

## Reflection and Transmission Interface

Consider the following setup:

- 2 semi-infinite strings of different densities $\rho_1 \neq \rho_2$
- Join 2 strings together. Assume tensions remain the same and 2 strings have equal tension: $T$

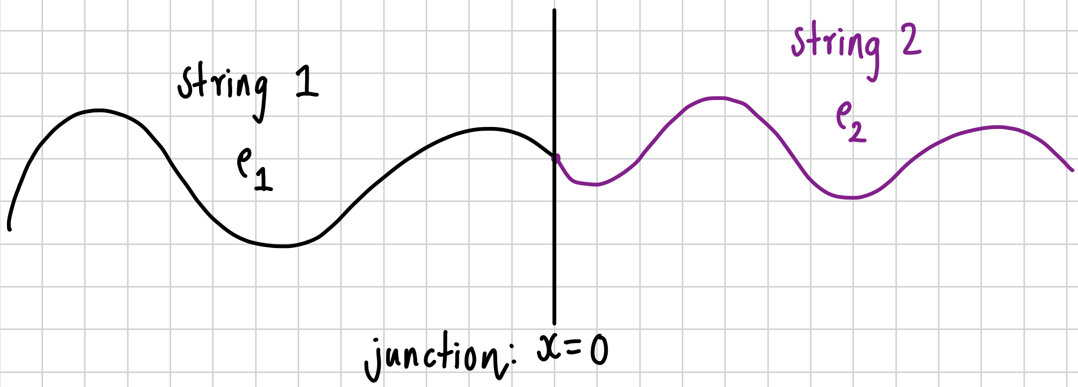



$\rho_1 \ll \rho_2$ : Second string is much heavier

A wave travelling on string 1, when it reaches the junction, some of the wave is transmitted and some is reflected.

The heavier string will offer a lot of restistance.

Analyzing mathematically




- String 1: along $-\infty < x < 0$
- String 2: along $0 < x < \infty$
- $\rho_1 \neq \rho_2$

We have a piecewise function

$$\boxed{y(x,t) = \begin{cases} f_1(x-c_1t) + g_1(x+c_1t) & x<0 \\ f_2(x-c_2t) + g_2(x+c_2t) & x>0 \end{cases}}$$

At $t=0$: We are looking at a right moving wave from left hand side. Mathematically

$$\begin{cases} f_1(x) \approx 0 & \forall x>0 \\ g_1(x) \approx 0 & \forall x<0 \end{cases} \qquad \begin{cases} f_2(x) \approx 0 & \forall x>0 \\ g_2(x) \approx 0 & \forall x>0 \end{cases}$$

Note: that $c_2>0$ and evolution of second string happens for $0 \leq t < \infty$.
This allows us to fix

$$g_2(x) = 0 \quad \text{identically}$$

Notation:

- $f_I(x-c_1t)$: Incident wave, right moving
- $g_R(x+c_1t)$: Reflected wave, left moving
- $g_T(x-c_2t)$: Transmitted wave, right moving

Therefore piecewise solution becomes

$$\boxed{y(x,t) = \begin{cases} f_I(x-c_1t) + g_R(x+c_1t) & x<0 \\ f_T(x-c_2t) & x>0 \end{cases}}$$

Imposing continuity: $y \in C^1 \implies$ continuously once differentiable and following conditions

$$\lim_{x\to 0^+} [y(x,t) - y(-x,t)] = 0$$

$$\lim_{x\to 0^+} [y'(x,t) - y'(-x,t)] = 0$$

Plugging in, we get

$$f_I(-c_1t) + g_R(+c_1t) = f_T(-c_2t) \qquad (1)$$

$$f_I'(-c_1t) + g_R'(+c_1t) = f_T'(-c_2t) \qquad (2)$$

Solving (1) and substituting in $s=-c_2t$

$$f_T(s) = f_I\left(\frac{c_1 s}{c_2}\right) + g_R\left(-\frac{c_1 s}{c_2}\right)$$

Substituting this into (2), we get

$$f_T'(s) = f_I'\left(\frac{c_1 s}{c_2}\right) + g_R'\left(-\frac{c_1 s}{c_2}\right) = \frac{c_1}{c_2} f_I'\left(\frac{c_1 s}{c_2}\right) - \frac{c_1}{c_2} g_R'\left(-\frac{c_1 s}{c_2}\right)$$

$$\Longrightarrow \quad \underbrace{\left[+1 + \frac{c_1}{c_2}\right]}_{\frac{c_1+c_2}{2}} \times g_R'\underbrace{\left(\frac{-c_1 s}{c_2}\right)}_{\sigma = -\frac{c_1}{c_2} s} = \underbrace{\left[\frac{c_1}{c_2} - 1\right]}_{\frac{c_1-c_2}{c_2}} f_I'\left(\frac{c_1 s}{c_2}\right)$$

Substituting σ, we get

$$g_R'(\sigma) = \frac{c_1 - c_2}{c_2} f_I'(-\sigma) \qquad \sigma = -\frac{c_1}{c_2} s$$

Integrating the equation; setting constant of integration to 0

$$\boxed{g_R(\sigma) = -\frac{c_1 - c_2}{c_1 + c_2} f_I(-\sigma)}$$

and finding transmitted wave $f_T(s)$,

$$f_T(s) = f_I\left(\frac{c_1 s}{c_2}\right) - \frac{c_1 - c_2}{c_1 + c_2} f_I(-\sigma) \Longrightarrow \boxed{f_T(s) = \frac{2c_2}{c_1 + c_2} f_I\left(\frac{c_1 s}{c_2}\right)}$$

Therefore the final solution is

$$\boxed{y(x,t) = \begin{cases} f_I(x - c_1 t) + A_R f_I(-x - c_1 t), & x < 0 \\ A_T f_I\left(\frac{c_1 (x - c_2 t)}{c_2}\right), & x > 0 \end{cases}}$$

where

$$\boxed{A_R = \frac{c_2 - c_1}{c_1 + c_2}}$$ REFLECTION AMPLITUDE

$$\boxed{A_T = \frac{2c_2}{c_1 + c_2}}$$ TRANSMISSION AMPLITUDE

## Limiting cases

- When $\rho_1 = \rho_2 \implies \sqrt{\frac{T}{\rho_1}} = \sqrt{\frac{T}{\rho_2}} \implies c_1 = c_2$

$$A_R = 0, \quad A_T = 1, \quad y(x,t) = f_I(x - c_1 t) \quad \forall x \in \mathbb{R} \quad \text{(right moving)}$$

- Suppose $\rho_1 \ll \rho_2$ ; right string is much heavier than the left one.

$$\rho_1 \ll \rho_2 \implies \frac{1}{\rho_1} \gg \frac{1}{\rho_2} \implies c_1 \gg c_2$$

Therefore

$$A_R = \lim_{\frac{c_1}{c_2} \to \infty} \frac{c_2 - c_1}{c_1 + c_2} = \lim_{\frac{c_1}{c_2} \to \infty} \left( \frac{\frac{c_1}{c_2} - 1}{\frac{c_2}{c_1} + 1} \right) \approx \frac{0 - 1}{0 + 1} = -1$$

Another way of looking at this is

$$A_R = \frac{c_2 - c_1}{c_1 + c_2} \text{ and } c_1 \gg c_2 \implies A_R \approx \frac{-c_1}{c_1} = -1$$

as $c_1$ dominates $c_2$ so $c_1 + c_2 \approx c_1$ and $c_2 - c_1 \approx -c_1$

Similarly

$$\boxed{A_T \approx 0} \qquad \boxed{A_R \approx -1}$$

Here heavy string as as a fixed point reflecting almost completely the incoming wave

- Consider $\rho_1 \gg \rho_2 \implies c_1 \ll c_2$

Here $$A_R = \lim_{\frac{c_2}{c_1} \to \infty} \left( \frac{1 - c_1/c_2}{1 + c_1/c_2} \right) = 1$$

$$A_T = \frac{2c_2}{c_1 + c_2} \approx 2\frac{c_2}{c_2} = 2$$

Therefore

$$\boxed{\begin{array}{c} A_R \approx 1 \\ A_T \approx 2 \end{array}}$$

## D'Alembert Wave Equation for Dirichlet Boundary

$$y(x,t) = f(x-ct) - f(-x-ct) \qquad (g(s) = -f(-s))$$

$$y_0(x) = f(x) - f(-x) \qquad v_0(x) = -cf'(x) + cf'(-x)$$

$$v_0(x) = -cf'(x) + cf'(-x) \implies \frac{d}{dx}\left(f'(x) - f'(-x)\right) = -\frac{v_0}{c}$$

$$\implies f(x) + f(-x) = -\frac{1}{c}\int_0^x v_0(s)\,ds$$

$$f(x) = \frac{1}{2}y_0(x) - \frac{1}{2c}\int_0^x v_0(s)\,ds$$

$$y(x,t) = f(x-ct) - f(-x-ct)$$

$$y(x,t) = \frac{y_0(x-ct) - y_0(-x-ct)}{2} - \frac{1}{2c}\left[\int_0^{x-ct} v_0(s)\,ds + \int_{-x-ct}^{0} v_0(s)\,ds\right]$$

$$= \frac{y_0(x-ct) - y_0(-x-ct)}{2} - \frac{1}{2c}\int_{-x-ct}^{x-ct} v_0(s)\,ds$$

$$\implies \boxed{y(x,t) = \frac{y_0(x-ct) - y_0(-x-ct)}{2} - \frac{1}{2c}\int_{-x-ct}^{x-ct} v_0(s)\,ds}$$

## D'Alembert Wave Equation for Dirichlet Boundary

$$y(x,t) = f(x-ct) + f(-x-ct)$$

Similarly to above

$$y_0(x) = f(x) + f(-x) \qquad v_0(x) = -cf'(x) - cf'(-x)$$

$$v_0(x) = -cf'(x) - cf'(-x) \implies f'(x) + f'(-x) = -\frac{v_0(x)}{c}$$

$$\implies f(x) - f(-x) = -\frac{1}{c}\int_0^x v_0(s)\,ds$$

$$f(x) = \frac{1}{2}y_0(x) - \frac{1}{2c}\int_0^x v_0(s)\,ds$$

$$\boxed{y(x,t) = \frac{y_0(x-ct) + y_0(-x-ct)}{2} - \frac{1}{2c}\int_0^{x-ct} v_0(s)\,ds - \frac{1}{2c}\int^{-x-ct} v_0(s)\,ds}$$

# Solution of Bernoulli: FINITE STRINGS

String is finite $\Longrightarrow$ 2 boundary conditions

$$\boxed{\partial_t^2 y(x,t) = c^2 \partial_x^2 y(x,t)}$$

**Seperation of variables**

**Ansatz:** $y(x,t) = X(x)T(t)$

Differentiating and substituting into wave equation

both sides are equal but depend on different variables

$$X(x)T''(t) = c^2 X''(x)T(t) \Longrightarrow \frac{1}{c^2}\frac{T''(t)}{T(t)} = \frac{X''(x)}{X(x)} = -k^2$$ constant

This is in **seperated form:**

- left hand only depends on $x$
- right hand only depends on $t$

Since equation must be equal and hold for all values of $x$ and $t$, both sides are equal to a constant

$$\Longrightarrow \frac{X''(x)}{X(x)} = \frac{1}{c^2}\frac{T''(t)}{T(t)} = -k^2$$

$$\Longrightarrow \begin{cases} X''(x) = -k^2 X(x) \\ T''(t) = -k^2 c^2 T(t) \end{cases}$$

The general solution is therefore

$$\boxed{\begin{aligned} X(x) &= A\cos(kx) + B\sin(kx) \\ T(t) &= F\cos(kct) + G\sin(kct) \end{aligned}}$$

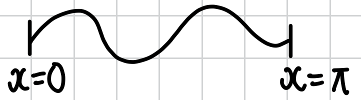
And therefore the wave equation takes form

$$\boxed{y(x,t) = \big(A\cos(kx) + B\sin(kx)\big)\big(F\cos(kct) + G\sin(kct)\big)}$$

# Finite Strings: standing waves and superpositions

## 2 Dirichlet Conditions: D-D condition

Finite string on interval $[0,\pi]$, $x=0$ and $x=\pi$ being fixed ends



$$\Longrightarrow \begin{cases} y(0,t)=0 \\ y(\pi,t)=0 \end{cases} \quad \forall t$$

Using seperated form, these conditions are satisfied if

$$X(0)=0 \qquad X(\pi)=0$$

Therefore

$$\begin{cases} X(0)=0 \\ X(\pi)=0 \end{cases} \Longrightarrow \begin{cases} A=0 \\ A+B\sin(k\pi)=0 \end{cases}$$

To avoid trivial solution $A=B=0$, set $B\neq 0$.

$$B\sin(k\pi)=0 \text{ and } B\neq 0 \Longrightarrow \sin(k\pi)=0$$
$$\Longrightarrow k\in\mathbb{Z}$$

Therefore

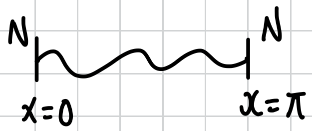
$$\boxed{A=0, \quad k\in\mathbb{Z}}$$

Therefore for D-D string

$$\boxed{y_k^{D\text{-}D}(x,t)=\sin(kx)\left(F_k\cos(kct)+G_k\sin(kct)\right) \qquad \forall\, k\in\mathbb{Z}}$$

where $F_k=BF$, $G_k=BG$

## 2 Neumann Conditions: N-N condition



$$\begin{cases} \partial_x y(x,t)\,|_{x=0}=0 \\ \partial_x y(x,t)\,|_{x=\pi}=0 \end{cases}$$

Using seperated form, these conditions are satisfied if

$$X'(0)=0 \qquad X'(\pi)=0$$

Therefore

$$\begin{cases} X'(0) = 0 \\ X'(\pi) = 0 \end{cases} \Longrightarrow \begin{cases} B = 0 \\ -Ak\sin(k\pi) + \overset{B=0}{\cancel{Bk\sin(k\pi)}} = 0 \end{cases} \qquad k \in \mathbb{Z}$$

To avoid trivial solution $A = B = 0$, set $A \neq 0$.

$$Ak\sin(k\pi) = 0 \text{ and } A \neq 0 \Rightarrow k\sin(k\pi) = 0$$
$$\Rightarrow k \in \mathbb{Z}$$

Therefore

$$\boxed{B = 0, \quad k \in \mathbb{Z}}$$

Therefore for D-D string

$$\boxed{y_k^{N-N}(x,t) = \cos(kx)\left(F_k \cos(kct) + G_k \sin(kct)\right) \qquad \forall\, k \in \mathbb{Z}}$$

Both $y^{N-N}$ and $y^{D-D}$ are standing waves. They don't travel but vibrate in place.

Superposition Principle

Superposition Principle

It states

> For all linear systems, linear combination of any number of solutions is a solution

Mathematically, given a linear system

$$LX = 0,$$

with solutions $\{X_i\}_{i=1}^{m}$ for some $m \geq 1$, then any linear combination

$$Y = \sum_{i=1}^{M} \alpha_i X_i, \qquad \alpha_i \in \mathbb{C}$$

is still a solution

$$L(Y) \equiv 0$$

Here: 1) $L$ is a matrix and $X$ a vector

2) $L$ is a differential operator and $X$ a function

Therefore

$$\boxed{\begin{array}{l} \text{if } y_1(x,t),\ y_2(x,t) \text{ solve wave equation then} \\ \qquad y_3(x,t) = \alpha y_1(x,t) + \beta y_2(x,t) + \gamma \\ \text{is a solution } \forall \alpha, \beta, \gamma \in \mathbb{C} \end{array}}$$

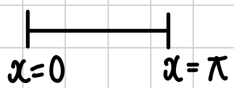
For example for D-D boundary problem, the most general solution is

$$y_K^{D\text{-}D}(x,t) = \sum_{K=1}^{\infty} \sin(Kx)\big(F_K \cos(Kct) + G_K \sin(Kct)\big) \qquad (*)$$

## Initial Value Problem

Choosing initial shape $y_0(x)$, initial velocity $v_0(x)$

Take a string at rest




$\xrightarrow{\text{pluck}}$

Initial Conditions

$$y(x,t=0) = y_0(x) \qquad \partial_t y(x,t=0) = v_0(x)$$

Assuming D-D boundary condition and substituting,

$$y_0(x) = \sum_{K=1}^{\infty} F_K \sin(Kx) \qquad\qquad v_0(x) = \sum_{K=1}^{\infty} KcG_K \sin(Kx)$$

To find co-efficients from the sums, we use orthogonality relations

$$\boxed{\frac{2}{\pi}\int_0^{\pi} \sin(Kx)\sin(\ell x)\,dx = \delta_{K\ell} = \begin{cases} 1 & K=\ell \\ 0 & K\neq\ell \end{cases}}$$

Hence we get

$$\frac{2}{\pi}\int_0^{\pi} y_0(x)\sin(\ell x)\,dx = \frac{2}{\pi}\int_0^{\pi} dx \sum_{K=1}^{\infty} F_K \sin(Kx)\sin(\ell x)$$

supposing sum converges $\displaystyle = \frac{2}{\pi}\sum_{K=1}^{\infty}\int_0^{\pi} dx\, F_K \sin(Kx)\sin(\ell x)$

$$= \sum_{k=1}^{\infty} F_k \frac{2}{\pi}\int_0^{\pi} dx \sin(kx)\sin(\ell x)$$

$$= \sum_{k=1}^{\infty} F_k \delta_{k\ell}$$

$$= F_\ell$$

Playing the same game

$$\frac{2}{\pi}\int_0^{\pi} V_0(x)\sin(\ell x)\, dx = \frac{2}{\pi}\int_0^{\pi} dx \sum_{k=1}^{\infty} kcG_k \sin(kx)\sin(\ell x)$$

suppossing sum converges $$= \sum_{k=1}^{\infty} \frac{2}{\pi}\int_0^{\pi} dx\, kcG_k \sin(kx)\sin(\ell x)$$

$$= \sum_{k=1}^{\infty} kcG_k \frac{2}{\pi}\int_0^{\pi} dx \sin(kx)\sin(\ell x)$$

$$= \sum_{k=1}^{\infty} kcG_k \delta_{k\ell}$$

$$= \ell c G_\ell$$

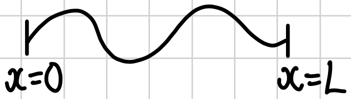
Therefore co-efficients are

$$\boxed{F_k = \frac{2}{\pi}\int_0^{\pi} y_0(x)\sin(kx)\, dx \qquad G_k = \frac{2}{\pi k c}\int_0^{\pi} V_0(x)\sin(kx)\, dx}$$

## String of generic length $x=L$

### 2 Dirichlet Conditions: D-D condition

Finite string on interval $[0,L]$, $x=0$ and $x=L$ being fixed ends




$$\Longrightarrow \begin{cases} y(0,t)=0 \\ y(L,t)=0 \end{cases} \quad \forall t$$

Using seperated form, these conditions are satisfied if

$$X(0)=0 \qquad X(\pi)=0$$

Therefore

$$\begin{cases} X(0)=0 \\ X(L)=0 \end{cases} \Longrightarrow \begin{cases} A=0 \\ A+B\sin(kL)=0 \end{cases}$$

To avoid trivial solution $A=B=0$, set $B\neq 0$.

$$B\sin(k\pi)=0 \text{ and } B\neq 0 \Rightarrow \sin(kL)=0$$

$$\Longrightarrow kL=n\pi$$

Therefore

$$\boxed{A=0, \quad k=\frac{n\pi}{L} \qquad n\in\mathbb{Z}}$$

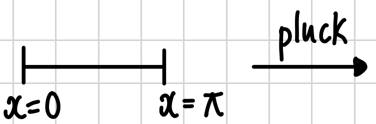
Therefore by superposition principle

$$y(x,t)=\sum_{n=0}^{\infty}\sin\left(\frac{n\pi x}{L}\right)\left[F_n\cos\left(\frac{n\pi}{L}ct\right)+G_n\sin\left(\frac{n\pi ct}{L}\right)\right]$$

### Initial Value Problem

Choosing initial shape $y_0(x)$, initial velocity $v_0(x)$

Take a string at rest




Initial Conditions

$$y(x,t=0)=y_0(x) \qquad \partial_t y(x,t=0)=v_0(x)$$

Using fact that

$$\frac{2}{L}\int_0 \sin\frac{n\pi x}{L}\sin\frac{m\pi x}{L}=\delta_{mn} \Longrightarrow$$

$$\boxed{\begin{aligned} F_n &= \frac{2}{L}\int_0^L y(x,0)\sin\left(\frac{n\pi x}{L}\right)dx \\ G_n &= \frac{2}{n\pi c}\int_0^L \frac{\partial}{\partial t}y(x,0)\sin\frac{n\pi x}{L}\,dx \end{aligned}}$$

# 2. Energy and Harmonics

## Harmonic Waves

As seen earlier, wave equation $\partial_t^2 y - c^2\partial_x^2 y = 0$ has solutions of form

$$y(x,t) = A\cos(k(x-ct)) + B\sin(k(x-ct)) + C\cos(k(x+ct)) + D\sin(k(x+ct))$$

This is a specific instance of the harmonic wave

$$\boxed{h(x,t) = a\cos\left[kx - \omega t + \phi\right]}$$

HARMONIC WAVES

- $a$ : Amplitude
- $k$ : Angular wave number
- $\omega$ : Angular frequency
- $\phi$ : phase

$$\boxed{\hat{k} = \frac{k}{2\pi}}$$

$$\boxed{\nu = \frac{\omega}{2\pi}}$$

The harmonic wave is also written as

$$\boxed{h(x,t) = a\cos(2\pi(\hat{k}x - \nu t) + \phi)}$$

**Dimensions**

- $[a] = h$
- $[k] = L^{-1}$
- $[\omega] = T^{-1}$
- $[\phi] = 1$

From the definition of $h(x,t)$, it is clear that

$$\boxed{\begin{aligned} &\text{Period: } P = \frac{1}{\nu} \\ &\text{Wave length: } \lambda = \frac{1}{\hat{k}} \end{aligned}}$$

## Properties of harmonic waves

- Since all constants of $h(x,t)$ are real,

$$\max_{x \text{ or } t} h(x,t) = a$$

So $a$ is the maximal displacement from $x$ axis

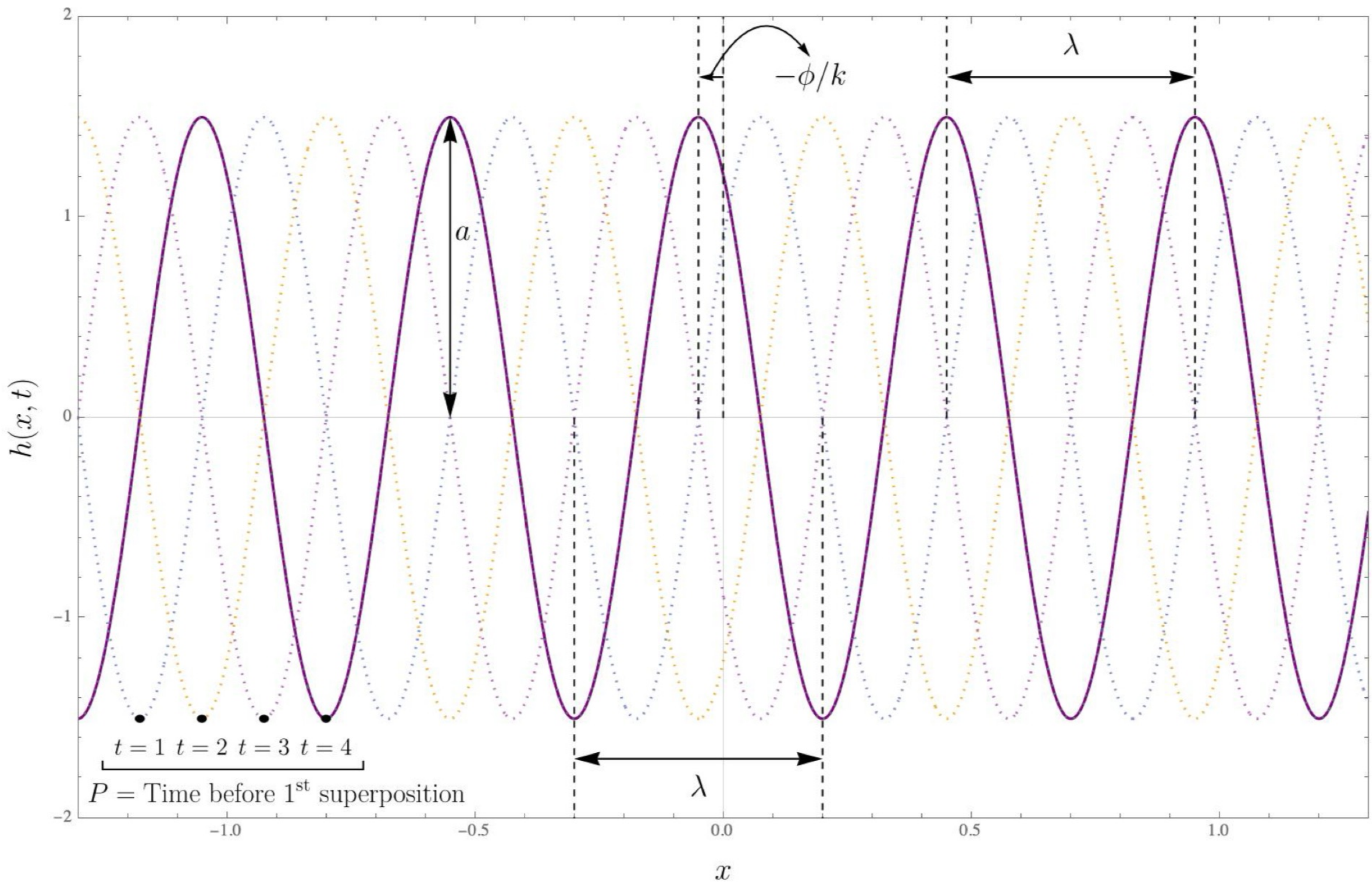
  - Nodes $h(x,t) = 0$
  - peaks $h(x,t) = a$
  - troughs $h(x,t) = -a$

- Since harmonic wave being sinusoid, it is periodic in $x$

$$h(x + n/\hat{k}, t) = h(x,t) \qquad \forall n \in \mathbb{Z}$$

The spatial period is therefore the $\boxed{\text{wavelength } \lambda = \frac{1}{\hat{k}}}$ : distance between 2 peaks

## Drawing a plot at $t = t_0$



- Similarly the wave $h(x,t)$ is periodic in $t$

$$h(x, t+n/\nu) = h(x,t) \quad \forall n \in \mathbb{Z}$$

This period is $\boxed{P = \frac{1}{\nu}}$ : The time that elapses from a reference instant $t_0$ before the $(x,h)$ plot of the wave superimposes itself for the first time

- frequency $\nu$: The number of times wave plot $(x,h)$ superimposes itself in a unit time interval $t \in (0,1]$

- Phase $\phi$ measures

  angular wave number: The displacement of crest closest to the reference point $x=0$ at reference time $t=0$.

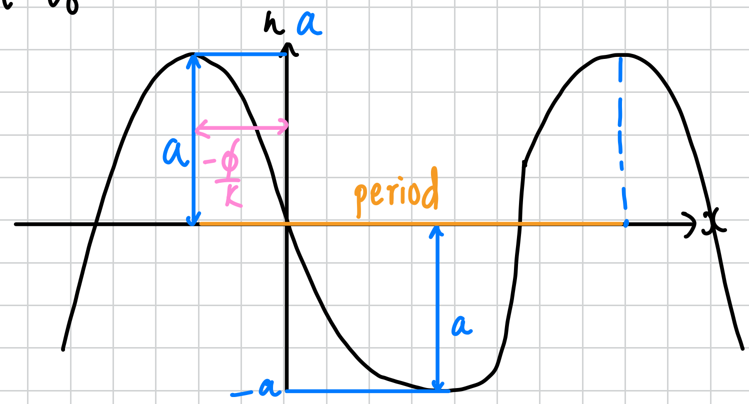
i.e.

$$h(\tilde{x}, 0) = a \implies \boxed{\phi = -k\tilde{x}}$$

- Wave/Phase speed:

$$\boxed{c = \lambda \nu = \frac{\nu}{\hat{k}} \equiv \frac{\omega}{k}} \qquad c = \text{constant}, \quad c = c(k)$$

$t = t_0$




$a\cos(kx - \omega t + \phi)$

$a\cos(kx_1 + \phi) \quad kx_2 = 2\pi + kx_1$

$\shortparallel$

$a\cos(kx_2 + \phi) \implies x_2 - x_1 = \frac{2\pi}{k} = \lambda$

wavelength

- Wave number $\hat{k}$: number of crests in unit interval $(0,1]$
- Wavelength $\lambda$: distance b/w 2 peaks or troughs

## Complex Harmonic Waves

$$\boxed{\begin{array}{ll} H(x,t) = Ae^{i(kx-\omega t)} & A\in\mathbb{C},\ k,\omega\in\mathbb{R} \\ A = ae^{i\phi} & \end{array}}$$

Taking real part

$$Re[H(x,t)] = Re\,A\cos(kx-\omega t) - Im\,A\sin(kx-\omega t)$$

$$H(x,t) = ae^{i(kx-\omega t+\phi)} \implies Re[H(x,t)] = a\,Re\left(e^{i(kx+\omega t+\phi)}\right) = a\cos(kx-\omega t+\phi)$$

Using complex harmonic waves to find solution to wave equation

$$\partial_t^2 H(x,t) = c^2\partial_x^2 H(x,t) \implies -A\omega^2 = -Ac^2k^2$$

All complex harmonic waves with

$$\omega = \omega(k) = \pm ck \implies v = v(\hat{k}) = \pm c\hat{k}$$

is a valid solution

## Solving PDE's with harmonic waves

> Any linear homogeneous PDE with constant co-efficients admits solutions in form of complex harmonic wave

## Heat Equation

$$\boxed{\partial_t u(x,t) = \alpha\partial_x^2 u(x,t)} \qquad u: \text{temperature} \qquad \alpha>0: \text{thermal diffusivity}$$

Let $u(x,t) = H(x,t) = Ae^{i(kx-\omega t)}$

Differentiating and substituting

$$-i\omega u(x,t) = \alpha(-k^2)u(x,t) \implies \omega + i\alpha k^2 = 0$$

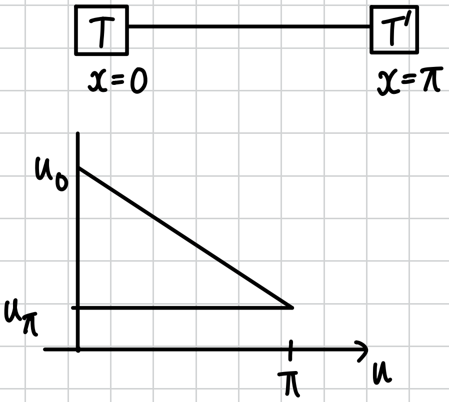
$$\implies \boxed{\omega = -i\alpha k^2}$$

Note: We defined $\omega\in\mathbb{R}$ but $\omega = -i\alpha k \in\mathbb{C} \Rightarrow$ solution extends to complex plane

Therefore

$$\boxed{u_k(x,t) = A_k e^{ikx-\alpha k^2 t} \qquad A \in \mathbb{C},\ k \in \mathbb{R}}$$

## Boundary Conditions




$$u(0,t) = T = u_0$$

$$u(\pi,t) = T' = u_\pi < u_0$$

Substituting boundary conditions

$$\begin{cases} Ae^{-\alpha k^2 t} = u_0 \\ Ae^{i\pi-\alpha k^2 t} = v_0 \end{cases}$$

cannot be solved

LHS has no time dependence

Use trick $\boxed{\partial_x^2 (a+bx) = 0}$

and therefore using $\boxed{u(x,t) = a + bx + U(x,t)}$ $\boxed{a = u_0 \quad b = \frac{u_\pi - u_0}{\pi} x}$

$$\partial_t u = \alpha \partial_x^2 u$$

$$\downarrow$$

$$\partial_t U = \alpha \partial_x^2 U$$

Therefore the most general solution is

$$\boxed{u(x,t) = a + bx + U(x,t)}$$

where $\boxed{U(0,t) = U(\pi,t) = 0}$ Dirichlet conditions

Since $U(0,t) = U(0,\pi) = 0$, $x$ appears in a sin function with $\sin(kz)$, $k \in \mathbb{Z}$

Observe that

$$u_k(x,t) + u_{-k}(x,t) = A_k e^{ikx-\alpha k^2 t} + A_{-k} e^{-ikx-\alpha k t^2}$$

Since $A_{-k}$ is just a constant define $A_{-k} = -A_k$. Then we get

$$u_k(x,t) + u_{-k}(x,t) = A_k e^{ikx-\alpha k^2 t} + A_{-k} e^{-ikx-\alpha k^2 t}$$
$$= 2i A_k e^{-\alpha k^2 t} \sin(kx)$$

$$\Longrightarrow \boxed{u_k(x,t) + u_{-k}(x,t) = 2i A_k e^{-\alpha k^2 t} \sin(kx)}$$

Also satisfies heat equation and boundary conditions $U(0,t) = U(\pi,t) = 0$

Hence by superposition principle define

$$\boxed{U(x,t) = \sum_{k=1}^{\infty} a_k e^{-\alpha k^2 t} \sin(kx)}$$

and hence

$$\boxed{u(x,t) = a + bx + U(x,t) = u_0 + \frac{u_\pi - u_0}{\pi} x + \sum_{k=1}^{\infty} a_k \sin(kx) e^{-\alpha k^2 t}}$$

Adding initial conditions

$$u(x,0) = u_0$$

The initial condition reads

$$u_0 + \frac{u_\pi - u_0}{\pi} x + \sum_{k=1}^{\infty} a_k \sin(kx) = u_\pi \qquad (*)$$

Remember the integrals

$$\frac{2}{\pi}\int_0^\pi dx \sin(kx)\sin(lx) = \delta_{kl}$$

and the trivial integrals

$$\frac{2}{\pi}\int_0^\pi dx \sin(lx) = \begin{cases} 4/(\pi l) & l \in 2\mathbb{Z}+1 \\ 0 & l \in 2\mathbb{Z} \end{cases}, \quad \frac{2}{\pi}\int_0^\pi dx\, x \sin(lx) = \frac{2}{l}(-1)^{l+1}$$

and integrating (∗) against $2/\pi \sin(lx)$, we get

$$\boxed{a_l = -2\,\frac{u_0 - u_\pi}{\pi l}}$$

We need to resum the series over $k$. For $t=0$,

$$\sum_{k=1}^{\infty} \frac{\sin(kx)}{k} = \frac{\pi - x}{2} \qquad \forall x \in [0,\pi]$$

Unfortunately, there is no closed for general $t$.

$$\boxed{u(x,t) = u_0 + \frac{u_\pi - u_0}{\pi} x - 2\frac{(u_0 - u_\pi)}{\pi} \sum_{k=1}^{\infty} \frac{\sin(kx)}{k} e^{-\alpha k^2 t}}$$

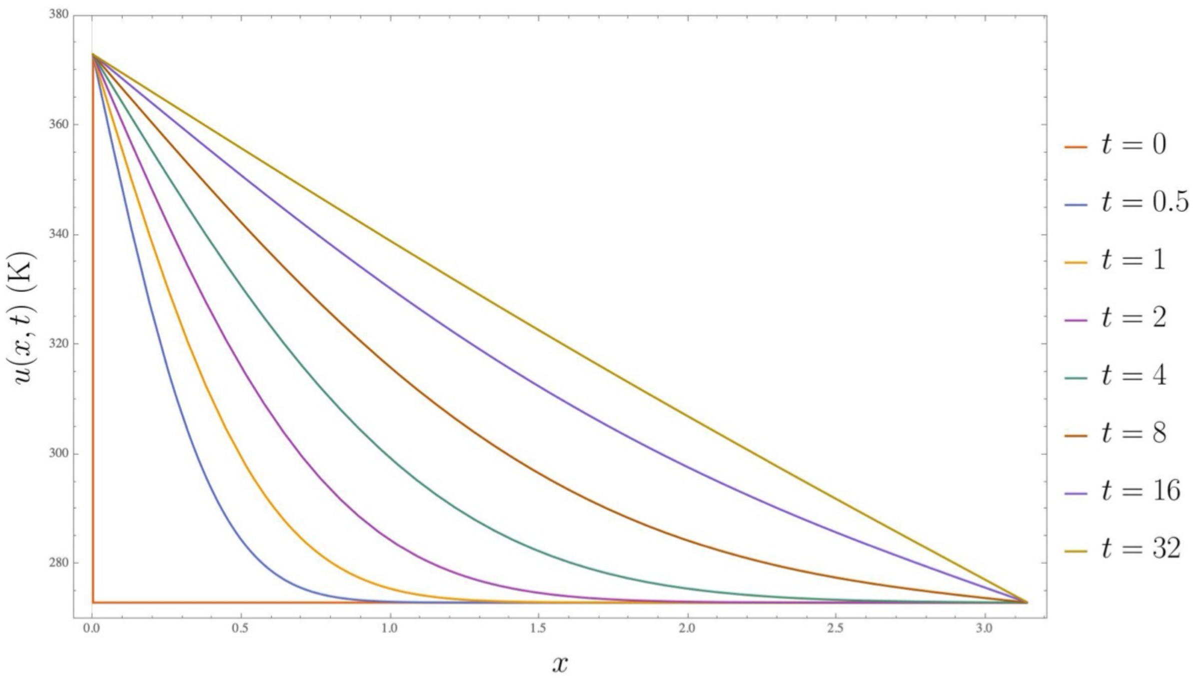
Taking the limit $t \to \infty$, the solution relaxes into a linear function

$$\lim_{t\to\infty} u(x,t) = u_0 - \frac{u_0 - u_\pi}{\pi} x$$

interpolating from the temperatures $u_0$ and $u_\pi$. On the other hand at $t=0$, the solution has a discontinuity at $x=0$ due to Dirichlet Conditions

$$u(x,0) = \begin{cases} u_0 & x=0 \\ u_\pi & x=\pi \end{cases}$$

taking an L-shape. The curves will smoothly deform with $t$ from $u(x,0)$ to $u(x,\infty)$




Plot of the solution (2.28) for $u_0 \simeq 373.15\text{K}$ (the boiling water point) and $u_\pi \simeq 273.15\text{K}$ (the freezing water point).

# Energy

## Energy

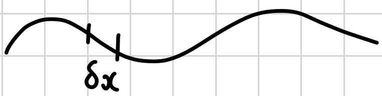
$$\boxed{\text{A wave is a disturbance in a medium that propogates energy}}$$

The total energy of a string is the sum of total kinetic energy and total potential energy

$$\boxed{\text{Total Energy} = E_{tot} = \underset{\substack{\text{total}\\ \text{KE}}}{K} + \underset{\substack{\text{total}\\ \text{PE}}}{V}}$$

## Energy density

Energy density is the energy of infitesmal part of a string between $x$ and $\delta x$




$$\boxed{\delta K(x,t) = \frac{1}{2} m v^2 = \frac{1}{2} \rho \delta x \left[\partial_t y(x,t)\right]^2}$$ Kinetic energy density

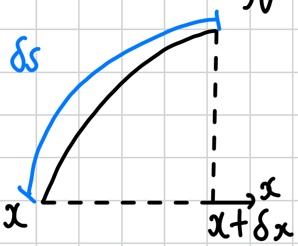
$$\boxed{m = \rho \delta x}$$

To obtain total KE, take limit $\delta x \to 0 \implies$ becomes integral

$$\boxed{K(t) = \int_0^L \mathcal{K}(x,t)\, dx}$$ Total Kinetic Energy

$$\boxed{\mathcal{K}(x,t) \equiv \frac{\rho}{2} \left[\partial_t y(x,t)\right]^2}$$

## Potential Energy




$$\delta s = \sqrt{1 + (\partial_x y(x,t)} \; \delta x \quad \text{arc length}$$

$$\cong \delta x \left[1 + \frac{1}{2}(\partial_x y(x,t) + O((\partial_x y)^4)\right]$$

Taylor Expansion

$\delta s << 1$ (small)

$$\boxed{\delta V(x,t) = T\delta s - T\delta x = \frac{\delta x T}{2}\left(\partial_x y(x,t)\right)^2}$$ Potential Energy density

To obtain total PE, take limit $\delta x \to 0 \Rightarrow$ becomes integral

Total PE

$$\boxed{V(t) = \int_0^L dx\, \mathcal{V}(x,t)} \qquad \boxed{\mathcal{V}(x,t) = \frac{T}{2}\left(\partial_x y(x,t)\right)^2}$$

Total Energy

$$\boxed{E(t) = K(t) + V(t) = \int_0^L dx\, \mathcal{E}(x,t)} \qquad \boxed{\mathcal{E}(x,t) = \frac{1}{2}\left[\rho(\partial_t y(x,t))^2 + T(\partial_x y(x,t))^2\right]}$$

Energy density

## Energy density of Example Waves

1) Right travelling wave : $f(x-ct)$

$$\begin{cases} \partial_x f(x-ct) = f'(x-ct) \\ \partial_t f(x-ct) = -c f'(x-ct) \end{cases}$$

$$\mathcal{E}(x,t) = \frac{f'(x-ct)^2}{2}\left[\rho c^2 + T\right] \qquad c = \sqrt{\frac{T}{\rho}}$$

$$= f'(x-ct)^2 T$$

$$E = \int_0^L dx\, \mathcal{E}(x,t) = T\int_0^L dx\, [f'(x-ct)]^2$$

2) Standing Waves: Consider D-D condition

$$y(x,t) = \sin(kx)\left(F\cos(kct) + G\sin(kct)\right)$$

$$= A\sin(kx)\cos(kct + \phi) \qquad \text{phasor addition}$$

$$A^2 = F^2 + G^2$$

$$\cos(\phi) = \frac{F}{\sqrt{F^2+G^2}}$$

Computing derivatives

$$\begin{cases} \partial_x y(x,t) = kA\cos(kx)\cos(kct) \\ \partial_t y(x,t) = -kcA\sin(kx)\sin(kct) + \phi \end{cases}$$

$$\varepsilon(x,t) = \frac{k^2A^2}{2}\left[\rho c^2 \sin^2(kx)\sin^2(kct+\phi) + T\cos^2(kx)\cos^2(kct+\phi)\right]$$

where

$$\boxed{\begin{aligned} \kappa(x,t) &= \frac{A^2k^2T}{2}\sin^2(kx)\sin^2(kct+\phi) \\ V(x,t) &= \frac{A^2k^2T}{2}\cos^2(kx)\cos^2(kct+\phi) \end{aligned}}$$

Remember integral

$$\int_0^\pi \sin^2(kx)\,dx = \frac{\pi}{2} \quad , \quad \int_0^\pi \cos^2 kx = \frac{\pi}{2}$$

We get

$$K(t) = \frac{A^2k^2T}{2}\sin^2(kct+\phi)\int_0^\pi \sin^2(kx)\,dx = \frac{A^2k^2T\pi}{4}\sin^2(kct+\phi)$$

$$V(t) = \frac{A^2k^2T}{2}\cos^2(kct+\phi)\int_0^\pi \cos^2(kx)\,dx = \frac{A^2k^2T\pi}{4}\cos^2(kct+\phi)$$

Adding the two terms, we get

$$\boxed{E(t) = K(t) + V(t) = \frac{A^2k^2T\pi}{4}}$$

## 3) Bichromatic wave

$$\boxed{\begin{aligned} y(x,t) &= y_k(x,t) + y_\ell(x,t) \\ &= A_k \sin(kx)\cos(kct+\phi) + A_\ell \sin(\ell x)\cos(\ell ct + \phi_\ell) \end{aligned}}$$

Also contains 2 fundamental frequencies $\omega_k = kc$, $\omega_\ell = \ell c$

Suppose $k \neq \ell$,

$$K(t) = \frac{\rho}{2}\int_0^L dx \left(\partial_t y(x,t) + \partial_t y_\ell(x,t)\right)^2$$

$$= K_k + K_\ell + \rho\int_0^L dx\, \partial_t y_k(x,t)\partial_t y_\ell(x,t)$$

Remember $\int_0^L \sin(kx)\sin(\ell x)\, dx = \frac{\pi}{2}\delta_{k\ell}$

Therefore if $k \neq \ell$,

$$\boxed{K = K_k + K_\ell}$$

Same holds for potential energy

$$\boxed{V = V_\ell + V_k}$$

Therefore

$$\boxed{E = E_k + E_\ell}$$

| |
|---|
| The total energy of a sum of standing wave is equal to sum of the individual standing wave energies |

We write this as

$$\boxed{E\left[\sum_k y_k(x,t)\right] = \sum_k E\left[y_k(x,t)\right]}$$

# Conservation Equation

Consider total energy

$$E_{tot} = \int_0^L dx\, \varepsilon(x,t) \Longrightarrow \frac{d}{dt} E_{tot} = \frac{d}{dt}\int_0^L dx\, \varepsilon(x,t)$$

Swap integral and derivative supposing integral converges (energy cannot be $\infty$)

$$\frac{d}{dt} E_{tot} = \int_0^L dx \frac{\partial}{\partial t} \varepsilon(x,t)$$

Differentiating energy density $\varepsilon(x,t)$,

$$\varepsilon(x,t) = \frac{1}{2}\left[\rho(\partial_t y(x,t))^2 + T(\partial_x y(x,t))^2\right]$$

we get

$$\begin{aligned}\partial_t \varepsilon(x,t) &= \rho\, \partial_t y(x,t) \partial_t^2 y(x,t) + T \partial_x y(x,t) \partial_x \partial_t y(x,t) \\ &= T\left[\partial_t y(x,t) \partial_x^2 y(x,t) + \partial_x y(x,t) \partial_x \partial_t y(x,t)\right]\end{aligned}$$

Note:

$$\partial_x\left[\partial_x y\, \partial_t y\right] = \partial_x^2 y\, \partial_t y + \partial_x y\, \partial_x \partial_t y$$

Hence

$$\boxed{\frac{\partial \varepsilon}{\partial t} = T\frac{\partial}{\partial x}\left[\frac{\partial}{\partial x} y(x,t) \frac{\partial}{\partial t} y(x,t)\right]}$$

Define energy flux as $\mathcal{F}$ as

$$\boxed{\mathcal{F}(x,t) = -T \partial_x y(x,t) \partial_t y(x,t)}$$ Energy flux

and we write $$\boxed{\partial_t \varepsilon(x,t) + \partial_x \mathcal{F}(x,t) = 0}$$ Conservation equation

$$\frac{dE}{dt} \equiv \int_{x_1}^{x_2} dx\, \partial_t \varepsilon(x,t) = -\int_{x_1}^{x_1} dx\, \partial_x \mathcal{F}(x,t) = \mathcal{F}(x_1,t) - \mathcal{F}(x_2,t)$$

# Boundary Conditions

$\frac{dE}{dt}$ may not be 0. It value depends on boundary conditions.

What the above equation is saying is the energy changes in time by the same amount the energy flows in/out from end points of the string

In Dirichlet and Neumann, string is studied in an isolated environment $\Rightarrow$ closed system $\Rightarrow dE/dt = 0$

- For D-D boundary

$$y(0,t) = y(L,t) = 0$$
$$\partial_t(0,t) = \partial_t y(L,t) = 0$$

$$F(x=0,t) = -T\partial_x y(x,t)\Big|_{x=0} \partial_t y(0,t) = 0$$

- For N-N boundary

$$\partial_x(x,t)\Big|_{x=0} = \partial_x y(x,t)\Big|_{x=L} = 0$$

For N-N, D-D, N-D, D-N,

$$\frac{dE}{dt} = 0$$

# 3. Bodies Vibrating in 3D

## Strings in 3D

$$\begin{cases} \partial_t^2 y(x,t) = c^2 \partial_t^2 y(x,t) \\ \partial_t^2 z(x,t) = c^2 \partial_x^2 z(x,t) \end{cases}$$

## Waves on a plane

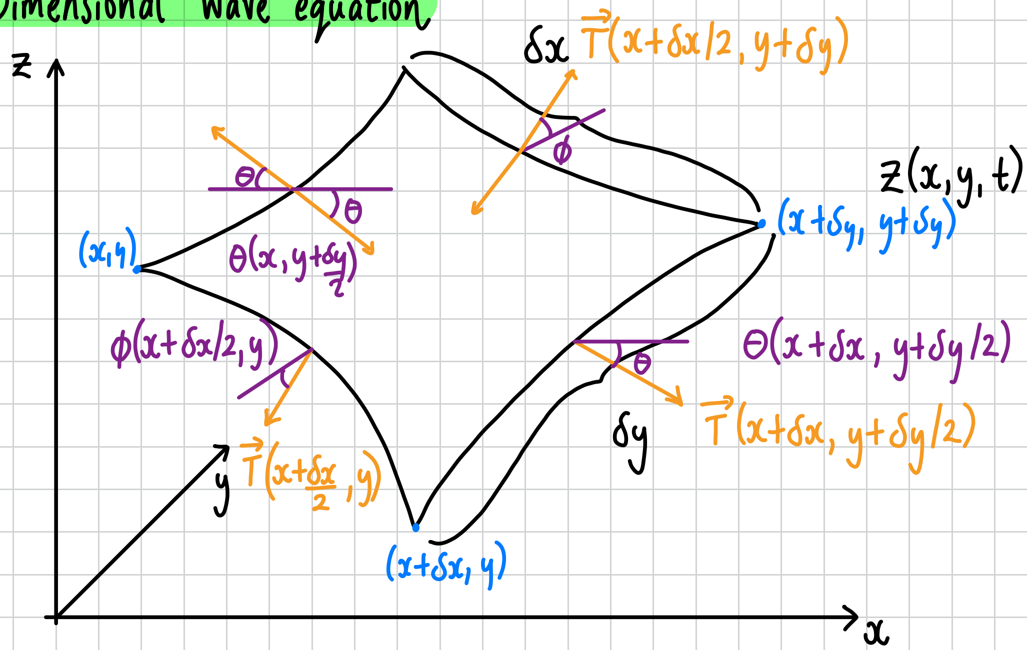
Consider an infinite 2-Dimensional Membrane of homogeneous density $\rho$

Equilibrium state is flat. Assume membrane is stretched with tension T. Each line segment will experience tension force along the line itself as the 1-D case

However, there will also be tension force acting in the direction perpendicular to the line.

Combinations of all the tensions will produce the total force.

## 2 Dimensional Wave equation




We make the following assumptions

- Membrane only traverses in z-direction.
- Tension remains constant and is the only force
- Angle between the horizontal plane z=0 and plane tangent to $z(y,x,t)$ is small.

$$\partial_x z(x,y,t) \ll 1 \qquad \partial_y z(x,y,t) \ll 1 \qquad \forall x,y,t$$

- $\vec{T}(x,y) = \underbrace{|\vec{T}|}_{\text{constant}} \vec{v}$

Hence

$$F_z = T\delta x\delta y\left[\sin\theta(x+\delta x, y+\delta y/2) - \sin\theta(x, y+\delta y/2) + \sin\phi(x+\delta x/2, y+\delta y/2) - \sin\phi(x+\delta x/2, y)\right]$$

- $\theta(x,y)$ is the angle made by vector $\vec{T}(x,y)$ along the $x$ direction and horizontal plane
- $\theta(x,y)$ is the angle made by vector $\vec{T}(x,y)$ along the $y$ direction and vertical plane

$$\theta(x,y) \sim \tan[\theta(x,y)] = \partial_x z(x,y,t) \qquad \delta x << 1$$
$$\phi(x,y) \sim \tan[\phi(x,y)] = \partial_y z(x,y,t) \qquad \delta y << 1$$

Using small angle approximation,

$$F_z(x,y,t) = T\delta x\delta y\left[\partial_x z(x,y,t) + \partial_y z(x,y,t)\right]$$

Applying Newton's second law $F_z(x,y,t) = m\partial_t^2 z(x,y,t) = \rho\delta x\delta y\partial_t^2 z(x,y,t)$

$$\partial_t^2 z(x,y,t) = c^2\left[\partial_x^2 z(x,y,t) + \partial_y^2 z(x,y,t)\right]$$

Therefore, we have

$$\boxed{\partial_t^2 z(x,y,t) = c^2\underline{\nabla}^2 z(x,y,t)}$$

2D WAVE EQUATION

Note:

$\underline{\nabla}^2$ is the Laplacian operator

$$\underline{\nabla}^2 = \partial_x^2 + \partial_y^2$$

Wave equation in dimension D

$$\boxed{\partial_t^2 f(\underline{x},t) = c^2\underline{\nabla}^2 f(\underline{x},t)}$$

Energy of a membrane

Energy density

$$\boxed{\varepsilon(x,y,t) = \frac{\rho}{2}\left(\partial_t z(x,y,t)\right)^2 + \frac{T}{2}\left[\left(\partial_x z(x,y,t)\right)^2 + \left(\partial_y z(x,y,t)\right)^2\right]}$$

where

$$\boxed{\begin{aligned} \mathcal{K}(t) &= \frac{\rho}{2}\left(\partial_t z(x,y,t)\right)^2 \\ \mathcal{U}(t) &= \frac{T}{2}\left[\left(\partial_x z(x,y,t)\right)^2 + \left(\partial_y z(x,y,t)\right)^2\right] \end{aligned}}$$

Note:

Gradient: $\nabla f(x,y) = \begin{pmatrix} \partial_x f(x,y) \\ \partial_y f(x,y) \end{pmatrix}$

Divergence: $\nabla \cdot \vec{V} = \partial_x V_x(x,y) + \partial_y V_y(x,y)$

Laplacian: $\nabla^2 f = \nabla \cdot (\nabla f) = \partial_x^2 f(x,y) + \partial_y^2 f(x,y)$

Therefore using $\nabla$ operator, $\varepsilon(x,y,t)$ becomes

$$\boxed{\varepsilon(x,y,t) = \frac{\rho}{2}\left(\partial_t z(x,y,t)\right)^2 + \frac{T}{2}\left|\nabla z(x,y,t)\right|^2}$$

Computing time derivative

$$\partial_t \varepsilon(x,y,t) = \rho\, \partial_t z(x,y,t)\, \partial_t^2 z(x,y,t) + T \nabla z(x,y,t) \cdot \nabla \partial_t z(x,y,t)$$

using wave equation and $\rho c^2 = T$

$$\partial_t \varepsilon(x,y,t) = \rho\, \partial_t z(x,y,t)\, \partial_t^2 z(x,y,t) + T \nabla z(x,y,t) \cdot \nabla \partial_t z(x,y,t)$$
$$= -\nabla \cdot \left(-T \partial_t z(x,y,t)\, \nabla z(x,y,t)\right)$$

Define flux as

Flux $$\boxed{\vec{F}(x,y,t) \equiv -T \partial_t z(x,y,t)\, \nabla z(x,y,t) = -T \partial_t z(x,y,t) \begin{pmatrix} \partial_x z(x,y,t) \\ \partial_y z(x,y,t) \end{pmatrix}}$$

Conservation equation $$\boxed{\partial_t \varepsilon(x,t) + \nabla \cdot \vec{F}(x,y,t) = 0}$$

Energy of a membrane is

$$\boxed{E = \iint_R dA\, \varepsilon(x,y,t)}$$

$dA = dx\,dy$

area element

$$\boxed{\frac{dE}{dt} = \iint_R dA\, \partial_t \varepsilon(x,y,t) = -\iint_R dA\, \nabla \cdot \vec{F}(x,y,t) = -\oint_{\partial R} ds\, \hat{n} \cdot \vec{F}(x,y,t)}$$

by 2D Gauss' Theorem

## Plane Waves

We call $z(x,y,t)$ a 2-Dimensional plane wave if it varies only in a single direction on the plane.

Direction determined by unit vector $\hat{n} = (n_x \;\; n_y)^T = \begin{pmatrix} n_x \\ n_y \end{pmatrix}$

$$|\hat{n}| = 1$$

Hence, mathematically

$$\boxed{z(x,y,t) = z(\hat{n}\cdot\vec{x}, t) = z(n_x x + n_y y, t)}$$

Plane Wave Equation

$$\vec{x} = \begin{pmatrix} x \\ y \end{pmatrix}, \quad \xi = n_x x + n_y y$$

Checking if this satisfies 2D-Wave equation

$$\partial_t^2 z(\xi,t) - c^2\left(n_x^2 \partial_\xi^2 z(\xi,t) + n_y^2 \partial_\xi^2 z(\xi,t)\right) = 0$$

where $\xi = \hat{n}\cdot\vec{x}$

$$\boxed{\partial_t^2 z(\xi,t) - c^2 |\hat{n}|^2 \partial_\xi^2 z(\xi,t) = 0} \longrightarrow \boxed{z(x,y,t) = f(\hat{n}\cdot\vec{x} - ct) + g(\hat{n}\cdot\vec{x} + ct)}$$

$|\hat{n}|^2 = 1$

right moving

left moving

Hence for plane waves, 2D-Wave equation reduces to 1D Wave eqn along a specific direction

$$\boxed{z(x,y,t) = f(\hat{n}\cdot\vec{x} - ct) + g(\hat{n}\cdot\vec{x} + ct)}$$

2D dimensional plane waves

There exists a notion of 2-Dimensional harmonic plane wave.

$$\boxed{h(x,y,t) = e^{i(\vec{k}\cdot\vec{x} - \omega t)}}$$

$\vec{k}$ : wave vector

Harmonic plane wave solves the 2Dimensional wave equation iff dispersion relation is satisfied

$$\omega = \omega(\vec{k}) = c|\vec{k}| = c\sqrt{k_x^2 + k_y^2}$$

# Rectangular Membranes

Has domain

$$\boxed{D_{a,b} = \{(x,y) \in \mathbb{R}^2 \mid 0<x<a,\ 0<y<b\}}$$

We want to find solutions to the wave equation

$$[\partial_t^2 - c^2\partial_x^2 - c^2\partial_y^2]\, z(x,y,t) = 0 \qquad \forall (x,y) \in D_{ab}$$

Imposing Dirichlet Boundary

$$\begin{cases} z(0,y,t)=0 \\ z(a,y,t)=0 \end{cases} , \qquad \begin{cases} z(x,0,t)=0 \\ z(x,b,t)=0 \end{cases}$$

## Separation of Variables

Employ the following ansatz

$$z(x,y,t) = X(x)Y(y)T(t)$$

Substitution in the wave equation and dividing everything by $z(x,y,t)$

$$\frac{1}{c^2}\frac{T''(t)}{T(t)} = \frac{X''(x)}{X(x)} + \frac{Y''(y)}{Y(y)} = -k^2 \qquad \text{constant}$$

We split the wave equation into 2 pieces

$$T''(t) = -c^2k^2T(t)$$

$$\frac{X''(x)}{X(x)} = -k^2 - \frac{Y''(y)}{Y(y)} = -\mu^2 \qquad \text{new constant}$$

Performing a further split, we get 3 independent ODE's

$$\begin{cases} T''(t) = -c^2k^2T(t) \\ X''(x) = -\mu^2X(x) \\ Y''(y) = -\nu^2Y(y) \end{cases} \qquad k^2 = \mu^2 + \nu^2$$

Solving the ODE's

$$\boxed{\begin{aligned} X(x) &= A\cos(\mu x) + B\sin(\mu x) \\ Y(y) &= C\cos(\nu y) + D\sin(\nu y) \\ T(t) &= E\cos(kct) + F\sin(kct) \end{aligned}}$$

$A\ B\ C\ D\ E\ F\ \mu,\nu$ constants

## Boundary Conditions

Imposing boundary conditions

$$\boxed{\underline{z(0,y,t) = z(a,y,t)} = \underline{z(x,0,t) = z(x,b,t)} = 0}$$

We see that

* $z(0,y,t) = z(a,y,t) \implies \begin{cases} X''(0) = 0 \\ X''(a) = 0 \end{cases}$

$$\implies \begin{cases} A = 0 \\ B\sin(\mu a) = 0 \end{cases} \implies \sin(\mu a) = 0 \qquad B \neq 0$$

$$\implies \mu = \frac{\pi n}{a} \qquad n \in \mathbb{Z}$$

* $z(0,y,t) = z(a,y,t) \implies \begin{cases} Y''(0) = 0 \\ Y''(a) = 0 \end{cases}$

$$\implies \begin{cases} C = 0 \\ D\sin(\mu a) = 0 \end{cases} \implies \sin(\nu b) = 0 \qquad D \neq 0$$

$$\implies \nu = \frac{\pi n}{b} \qquad n \in \mathbb{Z}$$

Therefore we get the following solution

$$\boxed{z^D_{m,n}(x,y,t) = \sin\left(\frac{\pi}{a} m x\right) \sin\left(\frac{\pi}{b} n y\right) \left[F_{m,n} \cos(K_{m,n} ct) + G_{m,n} \sin(K_{m,n} ct)\right]}$$

Normal Modes

$$K_{m,n} = K = \sqrt{\left(\frac{\pi}{a} n\right)^2 + \left(\frac{\pi}{b} m\right)^2} \qquad \forall m,n \in \mathbb{Z}$$

$$F_{m,n},\ G_{m,n} \in \mathbb{R}$$

By superposition principle

$$\boxed{z^D(x,y,t) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} z^D_{m,n}(x,y,t)}$$

The constants $F_{m,n}$, $G_{m,n}$ determined using initial conditions

$$z(x,y,0) = z_0(x,y)$$

$$\partial_t z(x,y,t)\Big|_{t=0} = V_0(x,y)$$

Substituting, we find

$$z(x,y,0) = \sum_{n,m}^{\infty} F_{(m,n)} \sin\left(\frac{\pi}{a} m x\right) \sin\left(\frac{\pi}{b} n y\right) = z_0(x,y)$$

$$\partial_t z(x,y,t)\Big|_{t=0} = \sum_{n,m}^{\infty} G_{(m,n)} K_{m,n} c \sin\left(\frac{\pi}{a} m x\right) \sin\left(\frac{\pi}{b} n y\right) = V_0(x,y)$$

Recall integral

$$\frac{2}{L}\int_0^L dx \sin\left(\frac{\pi}{L} n x\right) \sin\left(\frac{\pi}{L} m x\right) = \delta_{mn} \qquad \forall\, m,n \in \mathbb{Z}$$

$$\Rightarrow \frac{2}{a}\int_0^a dx \sin\left(\frac{\pi}{a} m' x\right) z_0(x,y) = \sum_{n,m} F \sin\left(\frac{\pi}{b} n y\right) \frac{2}{a}\int_0^a dx \sin\left(\frac{\pi}{a} m x\right) \sin\left(\frac{\pi}{a} m' x\right)$$

$$= \sum_{m=1}^{\infty} F_{(m',n)} \sin\left(\frac{\pi}{b} n y\right)$$

Similarly

$$\boxed{F_{m',n'} = \frac{4}{ab}\int_0^a dx \int_0^b dy\, z_0(x,y) \sin\left(\frac{\pi}{a} m' x\right) \sin\left(\frac{\pi}{b} n' y\right)}$$

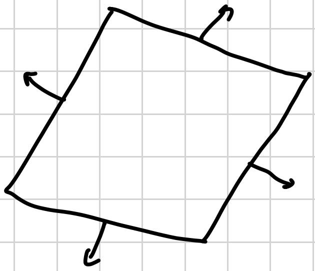
Applying initial velocity conditions, we get

$$\boxed{G_{m,n} = \frac{1}{K_{(m',n')}} \frac{4}{ab} \int_0^a \left[ \int_0^b \left( \sin\left(\frac{\pi}{a} m' x\right) \sin\left(\frac{\pi}{b} n' y\right) V_0(x,y) \right) dy \right] dx}$$

**Neumann boundary**: Free boundary

$$\hat{n}\cdot\underline{\nabla} z(x,y,t) = 0 \quad ; \qquad \hat{n} \text{ unit normal to } \partial D_{ab}$$

$$\begin{cases} \partial_x z(x,y,t)\Big|_{x=0,a} = 0 \\ \partial_y z(x,y,t)\Big|_{y=0,a} = 0 \end{cases}$$



Here we arrive at Normal Modes

$$\boxed{z^N_{m,n}(x,y,t) = \cos\left(\frac{\pi}{a}mx\right)\cos\left(\frac{\pi}{b}my\right)\left[F_{m,n}\cos(k_{m,n}ct) + G_{m,n}\sin(k_{m,n}ct)\right]}$$

## Circular Membranes

Has domain

$$\boxed{D_a = \{(x,y)\in\mathbb{R}^2 \mid x^2+y^2 < a\}}$$

**Dirichet Boundary**: $z(x,y,t) = 0 \qquad \forall (x,y)\in\partial D_a$

Applying polar co-ordinates

$$\begin{cases} x = r\cos\theta & r\in[0,\infty) \\ y = r\sin\theta & \theta\in[0,2\pi) \end{cases}$$

Note: in polar, $f(x,y) = f(r\cos\theta, r\sin\theta)$

$$\underline{\nabla}^2 = \partial_x^2 + \partial_y^2 = \partial_r^2 + \frac{1}{r}\partial_r + \frac{1}{r^2}\partial_\theta^2$$

Introduce

$$Z(r,\theta,t) = z(x(r,\theta),\ y(r,\theta),t)$$

Substituting into wave equation $\left[\partial_t^2 - c^2\underline{\nabla}^2\right]z = 0$ , we get

$$\frac{1}{c^2}\partial_t^2 Z = \partial_r^2 Z + \frac{1}{r}\partial_r Z + \frac{1}{r^2}\partial_\theta^2 Z$$

Applying seperation of variables

$$Z(r,\theta,t) = R(r)\Theta(\theta)T(t)$$

Same as before

$$T''(t) = -k^2 c^2 T(t)$$

$$\frac{R''(r)}{R(r)} + \frac{R'(r)}{rR(r)} + \frac{\Theta''(\theta)}{r^2\Theta(\theta)} = -k^2$$

The second equation can be further split

$$r^2\left(\frac{R''(r)}{R(r)} + \frac{R'(r)}{rR(r)} + k^2\right) = -\frac{\Theta''(\theta)}{\Theta(\theta)} = -n^2$$

And we get

$$\begin{cases} T''(t) = -k^2c^2T(t) \\ \Theta''(\theta) = -n^2\Theta(\theta) \longrightarrow \begin{array}{l} \Theta(\theta) = A\cos(n\theta) + B\sin(n\theta) \\ \Theta(\theta) = \Theta(\theta+2\pi) \Rightarrow n\in\mathbb{Z} \end{array} \\ R''(r) + \frac{1}{r}R'(r) + \left(k^2 - \frac{n^2}{r^2}\right)R(r) = 0 \end{cases}$$

Non-Dimensionalizing

$$\left[\frac{\frac{d^2R}{dr^2}}{R}\right] = \frac{[R]}{[r]}, \qquad [k] = \frac{1}{[r]}$$

Define $\rho = rk \Rightarrow [\rho] = 1$

$$\tilde{R}(\rho) = R(r(\rho)) \qquad \lambda = \frac{n}{k}$$

$$\boxed{\tilde{R}''(\rho) + \frac{1}{\rho}\tilde{R}'(\rho) + \left(1 - \frac{\lambda^2}{\rho^2}\right)\tilde{R}(\rho) = 0}$$

Bessel Equation